Building Random Trees from Blocks

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Joint work with

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The Probability Model

- We have a finite collection of unlabeled, rooted, nonplanar building blocks (trees) $\mathscr{C} = \{T_1, \ldots, T_k\}$, that occur with respective probabilities $p_1, \ldots, p_k$; $\sum_{j=1}^{k} p_j = 1$.

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- At step n, with probability p_i we choose the block T_i from $\mathscr{C}$, and we adjoin it to the tree $\mathcal{T}_{n-1}$ by choosing a "parent" node at random from $\mathcal{T}_{n-1}$ (all nodes from $\mathcal{T}_{n-1}$ being equally likely parents).

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- A special case: $\mathscr{C}$ consists of only one node; the block tree is isomorphic to the well-studied standard recursive tree.

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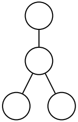

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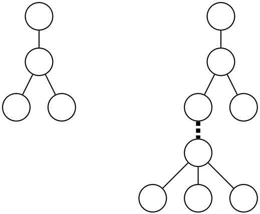

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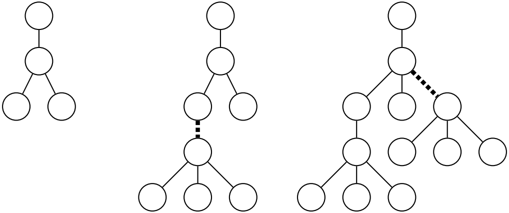

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Step 2: The second block occurs with probability $\frac{2}{3}$, and the parent is chosen with probability $\frac{1}{4}$;
Step 3: The third block occurs with probability $\frac{2}{3}$, and the parent is chosen with probability $\frac{1}{8}$.

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- Human Resources - the growth of large organizations

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  - Total path length of the blocks tree
  - Height of the blocks tree

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Theorem

Let L_n be the number of leaves in a random tree built from the building blocks $T_1, \ldots, T_k$, which are selected at each step with probabilities $p_1, \ldots, p_k$. Let $\mathbf{E}[\Lambda_{\mathscr{C}}]$ be the average number of leaves in the given collection, and $\mathbf{Var}[\Lambda_{\mathscr{C}}]$ be the variance. Then,

$$\frac{L_n - \frac{t\,\mathbf{E}[\Lambda_{\mathscr{C}}]}{t+1}\,n}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma^2_{\mathscr{C}}\right),$$

where

$$\sigma^2_{\mathscr{C}} := \Big(\frac{\mathbf{Var}[\Lambda_{\mathscr{C}}]}{t+2} + \frac{\mathbf{E}[\Lambda_{\mathscr{C}}](t+1-\mathbf{E}[\Lambda_{\mathscr{C}}])}{(1+t)^2(2+t)}\Big)t.$$

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- This will enable us to use the powerful theory of Pólya urns.
- Suppose T_i has ℓ_i leaves (and consequently it has $t - \ell_i$ internal nodes).
- Let $\Lambda_{\mathscr{C}}$ be number of leaves in a randomly chosen block, i.e., $\Lambda_{\mathscr{C}}$ has probability mass

$$P(\Lambda_{\mathscr{C}} = \ell) = \sum_j p_j,$$

where the sum is taken over all j such that block T_j has ℓ leaves.

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The replacement matrix for this urn:

$$\mathbf{A} = \left(\begin{array}{c|cc} & W & B \\ \hline W & \Lambda_{\mathscr{C}} - 1 & t - \Lambda_{\mathscr{C}} + 1 \\ B & \Lambda_{\mathscr{C}} & t - \Lambda_{\mathscr{C}} \end{array} \right)$$

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Row sum for this matrix is a constant! Balanced Urns!

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For balanced urns it is shown by [Athreya 1968] that

$$\frac{W_n}{n} \xrightarrow{\text{a.s.}} \lambda_1 v_1,$$

where λ_1 is the principal (largest real) eigenvalue of the average of the replacement matrix, and (v_1, v_2) is the corresponding left eigenvector of $\mathbf{E}[\mathbf{A}]$.

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and hence,

$$\frac{W_n}{n} \xrightarrow{\text{a.s.}} \frac{t}{t+1} \mathbf{E}[\Lambda_{\mathscr{C}}].$$

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Further we also notice that $\Re\lambda_2 < \frac{1}{2}\lambda_1$.
[Smythe 1996] showed that

$$\frac{W_n - \lambda_1 v_1}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2),$$

for some variance σ^2 and states that σ^2 is generally hard to compute, but we will obtain the exact variance of W_n.

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Solving these recurrences, we get

$$\begin{aligned} \mathbf{E}\big[W_n\big] &\sim \frac{t\,\mathbf{E}[\Lambda_{\mathscr{C}}]}{(t+1)}\, n + O\big(n^{-1/t}\big), \\ \mathbf{Var}[W_n] &\sim \Big(\frac{\mathbf{Var}[\Lambda_{\mathscr{C}}]}{t+2} + \frac{\mathbf{E}[\Lambda_{\mathscr{C}}](t+1-\mathbf{E}[\Lambda_{\mathscr{C}}])}{(1+t)^2(2+t)}\Big) tn \\ &\qquad + O(n^{1-\epsilon}) \\ &:= \sigma_{\mathscr{C}}^2\, n. \end{aligned}$$

Hence the theorem follows!

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Let D_n be the depth of the root of the nth inserted block in a random tree built from the building blocks $T_1, \ldots, T_k$, which are selected at each step with probabilities $p_1, \ldots, p_k$. Let $\mathbf{E}[\Delta_{\mathscr{C}}]$ be the average depth of a node in the given collection, and $\mathbf{Var}[\Delta_{\mathscr{C}}]$ be the variance of that depth. Then,

$$\frac{D_n - (\mathbf{E}[\Delta_{\mathscr{C}}] + 1) \ln n}{\sqrt{\ln n}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \mathbf{Var}[\Delta_{\mathscr{C}}] + (\mathbf{E}[\Delta_{\mathscr{C}}] + 1)^2\right).$$

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- $\Delta_{\mathscr{C}}$ (representing the depth of a parent node), which is completely determined by the structure of the trees in the collection;
- Each δ_n has the same random distribution as $\Delta_{\mathscr{C}}$.

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$$\begin{aligned} \mathbf{E}\big[e^{D_n u} \,|\, \mathcal{F}_{n-1}\big] &= \mathbf{E}\Big[\sum_{i=1}^{n-1} e^{(D_i+\delta_n+1)u} \frac{1}{n-1} \,\Big|\, \mathcal{F}_{n-1}\Big] \\ &= \frac{1}{n-1} \mathbf{E}\big[e^{(\delta_n+1)u}\big] \sum_{i=1}^{n-1} e^{D_i u}. \end{aligned}$$

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Solving this full history recurrence and after appropriate centering and scaling we get

$$\mathbf{E}\Big[\exp\Big(\Big(\frac{D_n - (\mathbf{E}[\Delta_{\mathscr{C}}] + 1)\ln n}{\sqrt{\ln n}}\Big)u\Big)\Big] \to e^{\frac{1}{2}\left(\mathbf{Var}[\Delta_{\mathscr{C}}] + (\mathbf{E}[\Delta_{\mathscr{C}}]+1)^2\right)u^2}.$$

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Therefore we have the theorem,

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$$\mathbf{E}\Big[\exp\Big(\Big(\frac{D_n - (\mathbf{E}[\Delta_{\mathscr{C}}]+1)\ln n}{\sqrt{\ln n}}\Big)u\Big)\Big] \to e^{\frac{1}{2}\left(\mathbf{Var}[\Delta_{\mathscr{C}}]+(\mathbf{E}[\Delta_{\mathscr{C}}]+1)^2\right)u^2}.$$

Therefore we have the theorem,

$$\frac{D_n - (\mathbf{E}[\Delta_{\mathscr{C}}]+1)\ \ln n}{\sqrt{\ln n}} \xrightarrow{\mathcal{D}} \mathcal{N}\big(0, \mathbf{Var}[\Delta_{\mathscr{C}}] + (\mathbf{E}[\Delta_{\mathscr{C}}]+1)^2\big).$$

Remark. The expressions for mean and variance are valid, even if $t = t(n)$ grows with n. However, in the asymptotic derivations of the central limit theorem we have to keep t relatively small, compared to n.

Total path length

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Theorem

Let X_n be the total path length of a tree built from the blocks of a collection $\mathscr{C}$. Then, there is an absolutely integrable random variable X, such that $X_n/n - \left(t + \mathbf{E}[\chi_{\mathscr{C}}]\right) H_n + t$ converges to X, both in $\mathcal{L}_2$ and almost surely.

Sketch of Proof.

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- Let $\chi_{\mathscr{C}}$ be a discrete random variable that assumes the value x_i, with probability $\sum_j p_j$,
  where the sum is taken over every block T_j with total path length x_i.
- For example:

Figure : A collection of building blocks of size 4, with probabilities $\frac{1}{3}$, and $\frac{2}{3}$, respectively.

$$P(\chi_{\mathscr{C}} = 5) = \frac{1}{3}, \qquad \text{and} \qquad P(\chi_{\mathscr{C}} = 3) = \frac{2}{3}.$$

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- If the nth block is adjoined to a node $v \in \mathcal{T}_{n-1}$, at depth $\tilde{D}(v)$ in the tree $\mathcal{T}_{n-1}$ then each node in the last inserted block appears at distance equal to $\tilde{D}(v) + 1$, plus its own depth in the last block.

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- So we have the following stochastic recurrence for $n \geq 2$:

$$\mathbf{E}[X_n \mid \mathcal{F}_{n-1}] = X_{n-1} + t\Big(\frac{1}{t(n-1)} \sum_{v \in \mathcal{T}_{n-1}} \tilde{D}(v) + 1\Big) + \mathbf{E}\big[\chi_{\mathscr{C}} \mid \mathcal{F}_{n-1}\big]$$

$$= X_{n-1} + \frac{X_{n-1}}{n-1} + t + \mathbf{E}[\chi_{\mathscr{C}}].$$

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- Solving we get

$$\mathbf{E}[X_n] = \big(t + \mathbf{E}[\chi_{\mathscr{C}}]\big)\, n\, H_n - nt \sim \big(t + \mathbf{E}[\chi_{\mathscr{C}}]\big) n \ln n,$$

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- We develop a separate recurrence for $\mathbf{E}[\check{D}_n^2]$ and solving for $\mathbf{E}[X_n^2]$ we get,

$$\mathbf{Var}[X_n] \sim \Big(t^2\big(\mathbf{E}[\Delta_{\mathscr{C}}^2] + 2(\mathbf{E}[\Delta_{\mathscr{C}}])^2 + 4\mathbf{E}[\Delta_{\mathscr{C}}] + 4\big) + \mathbf{E}[\chi_{\mathscr{C}}^2] + 4t\mathbf{E}[\chi_{\mathscr{C}}] \\ - \big(\mathbf{E}[\chi_{\mathscr{C}}] + t\big)^2\Big(2 + \frac{\pi^2}{6}\Big)\Big)n^2.$$

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Height

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Theorem

Let H_n be the height of a random tree built from the building blocks $T_1, \ldots, T_k$, which are selected at each step with probabilities $p_1, \ldots, p_k$. We then have

$$\frac{H_n}{\ln n} \xrightarrow{\text{a.s.}} e\big(\mathbf{E}[\Delta_{\mathscr{C}}] + 1\big).$$

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  - When a child of a node bursts into a block, it is its root that gets joined to a randomly chosen parent in the parent block.

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Hence
$H_n \geq (1 + \hat{\delta}_1) + (1 + \hat{\delta}_2) + \cdots + (1 + \hat{\delta}_{\hat{H}_n}) = \hat{H}_n + \sum_{i=1}^{\hat{H}_n} \hat{\delta}_i$,
where $\hat{\delta}_i$, for $i = 1, \ldots, \hat{H}_n$ are all independent.

Sketch of Proof.

- By the strong law of large numbers $\frac{1}{\hat{H}_n} \sum_{i=1}^{\hat{H}_n} \hat{\delta}_i \xrightarrow{\text{a.s.}} \mathbf{E}[\Delta_{\mathscr{C}}]$.

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- Then, the height of the blocks tree is bounded above:

$$H_n \le \max_{1\le i\le n}\left\{(1+\hat{\delta}_1^{(i)}) + (1+\hat{\delta}_2^{(i)}) + \cdots + (1+\hat{\delta}_{\hat{D}_i}^{(i)})\right\}$$

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$$\begin{aligned} H_n &\le \max_{1\le i\le n}\left\{\hat{D}_i + \hat{\delta}_1^{(i)} + \hat{\delta}_2^{(i)} + \cdots + \hat{\delta}_{\hat{D}_i}^{(i)}\right\} \\ &\le \max_{1\le i\le n}\hat{D}_i + \max_{1\le i\le n}\left\{\hat{\delta}_1^{(i)} + \hat{\delta}_2^{(i)} + \cdots + \hat{\delta}_{\hat{D}_i}^{(i)}\right\} \\ &\le \hat{H}_n + \max_{1\le i\le n}\left\{\hat{\delta}_1^{(i)} + \hat{\delta}_2^{(i)} + \cdots + \hat{\delta}_{\hat{H}_n}^{(i)}\right\}, \end{aligned}$$

where $\hat{\delta}_{\hat{D}_i+1}^{(i)}, \ldots, \delta_{\hat{H}_n^{(i)}}$ are additional independent random variable padded at the end to make all the expressions of the same length.

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&\le \hat{H}_n + \max_{1\le i\le n}\left\{\hat{\delta}_1^{(i)} + \hat{\delta}_2^{(i)} + \cdots + \hat{\delta}_{\hat{H}_n}^{(i)}\right\},
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&\le e + e\mathbf{E}[\Delta_{\mathscr{C}}] + o(1), \qquad \text{a.s.}
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Sketch of Proof.

$$H_n \le \max_{1\le i\le n}\left\{\hat{D}_i + \hat{\delta}_1^{(i)} + \hat{\delta}_2^{(i)} + \cdots + \hat{\delta}_{\hat{D}_i}^{(i)}\right\}$$
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$$\le e + e\mathbf{E}[\Delta_{\mathscr{C}}] + o(1), \qquad \text{a.s.}$$
$$\xrightarrow{\text{a.s.}} e\big(1 + \mathbf{E}[\Delta_{\mathscr{C}}]\big),$$

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$$
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&\xrightarrow{\text{a.s.}} e\big(1 + \mathbf{E}[\Delta_{\mathscr{C}}]\big),
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Combining the bounds the result follows!

Future Work and Challenges

- The distribution of the number of nodes of outdegree $j > 0$.

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- The joint distribution or atleast covariances of number nodes of outdegree i and j.
- We can not handle a collection of blocks of different sizes, which would be more general.
- The underlying structure here is a recursive tree. We could try to build other types of tree from blocks.

Thank You!
Questions are welcome!