Group divisible packing designs with block size 3: relationship to coverings

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Overview

1. CARLs or GDCDs
   - Definition
   - Bounds
   - Group divisible covering designs
   - Graph covering
   - Edge covering
   - Excess graph
2. PARLs or GDPD
   - Leave graph
   - Upper bounds
   - Transformable GDCD

Covering array with row limit (CARL)

$CARL_{\lambda=1}(N = 12; t = 2, k = 6, v = 2 : w = 4) :$

	1	2	3	4	5	6
1	0	0	–	1	–	0
2	0	–	1	1	1	–
3	0	1	0	–	–	1
4	0	–	0	0	0	–
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- N = size; number of rows
- goal: find minimal N

Central questions

- Optimal (minimal) size: $CARLN_\lambda(t, k, v : w)$

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- Construction of *CARLs* of optimal size

Bounds on size of CARLs

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If $w(k) = \Theta(k)$, *then*

$$CARLN(t, k, v\colon w(k)) = \mathrm{O}(\log k)$$

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Theorem

When $w(k) = c$, $c \in \mathbb{N}$,

$$CARLN(t, k, v : w) = \frac{\binom{k}{t}}{\binom{w}{t}} v^t (1 + \mathrm{o}(1))$$

Schönheim lower bound

Theorem

$$CARLN(t, k, v : w) \geq SB_\lambda(t, k, v : w)$$

$$SB_\lambda(t, k, v : w) = \left\lceil \frac{vk}{w} \left\lceil \frac{v(k-1)}{w-1} \cdots \left\lceil \frac{\lambda v(k-t+1)}{w-t+1} \right\rceil \cdots \right\rceil \right\rceil.$$

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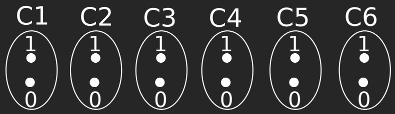
$|V| = 12$, $\mathcal{G} = \{C1, C2, \cdots, C6\}$



$CARL(12; 2, 6, 2 : 4)$

	1	2	3	4	5	6
1	0	0	–	1	–	0
2	0	–	1	1	1	–
⋮	⋮	⋮	⋮	⋮	⋮	⋮

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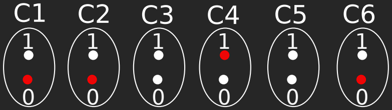

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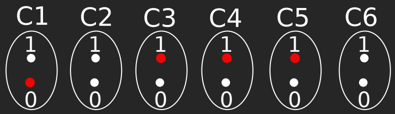

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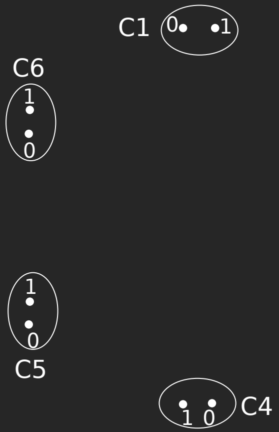
Edge covering

- graph covering of a $K_{\underbrace{g, g, \dots, g}_{u}}$ by K_k:



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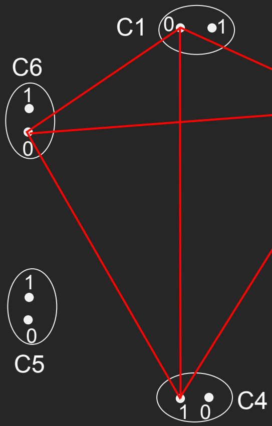

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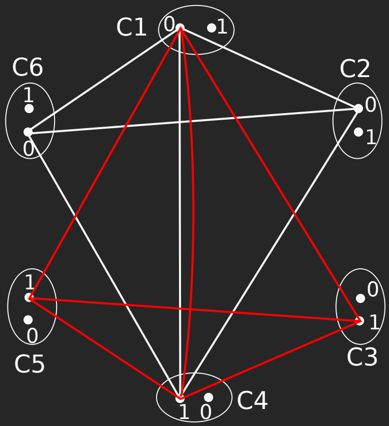

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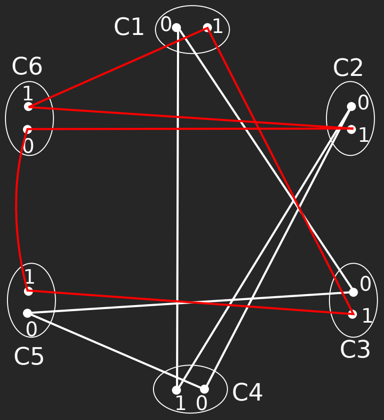
Excess graph when $t = 2$



$2C_6$

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- λ = index
- N = size; number of rows
- goal: find **maximal** N

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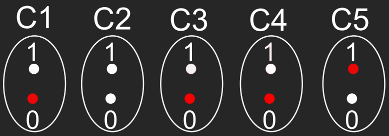

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Leave graph when $t = 2$

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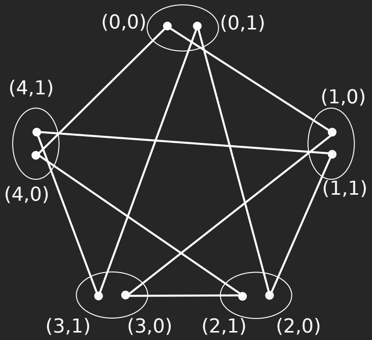

$2\ C_5$

Schönheim upper bound

Theorem

$$PARLN_\lambda(t,k,v:w) \leq \left\lfloor \frac{vk}{w} \left\lfloor \frac{v(k-1)}{w-1} \left\lfloor \cdots \left\lfloor \frac{v(k-t+2)}{w-t+2} \left\lfloor \frac{\lambda v(k-t+1)}{w-t+1} \right\rfloor \right\rfloor \cdots \right\rfloor \right\rfloor \right\rfloor .$$

Johnson upper bound for *PARL*

Theorem

If $vk\, PARLN_\lambda(t-1, k-1, v\colon w-1) \not\equiv 0 \pmod{w}$ *and for some* m *such that* $2 \le m \le t$,

$$PARLN_\lambda(t-1, k-1, v\colon w-1) = \frac{\binom{k-1}{m-1}}{\binom{w-1}{m-1}} v^{m-1} PARLN_\lambda(t-m, k-m, v\colon w-m),$$

then

$$PARLN_\lambda(t, k, v\colon w) \le \left\lfloor \frac{vkPARLN_\lambda(t-1, k-1, v\colon w-1) - m}{w} \right\rfloor.$$

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Theorem

Let $t \leq w$ and v be positive integers. Then,

$$PARLN(t, k, v : w) = \frac{\binom{k}{t}}{\binom{w}{t}} v^t (1 - o(1)).$$

Example of the transformation

- Start with an optimal $3 - GDCD$ of type 1^6

$V = \mathbb{Z}_6$
$\mathcal{G} = \{\{i\} : i \in \mathbb{Z}_6\}$
$\mathcal{B}$:

$$\begin{array}{l}\{0, \; 1, \; 2\}\\ \{1, \; 2, \; 5\}\\ \{0, \; 3, \; 5\}\\ \{0, \; 4, \; 5\}\\ \{1, \; 3, \; 4\}\\ \{2, \; 3, \; 4\}\end{array}$$

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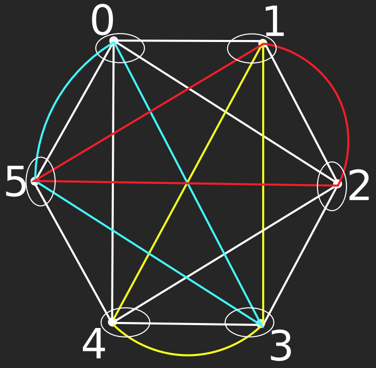

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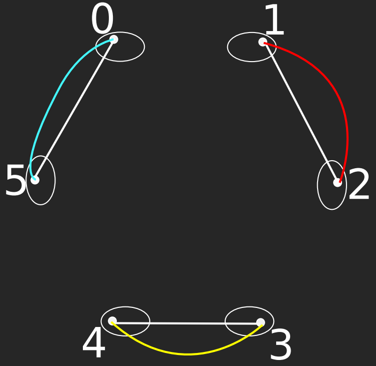
Excess coverage:



Example of the transformation

- Remove blocks which contribute to the excess

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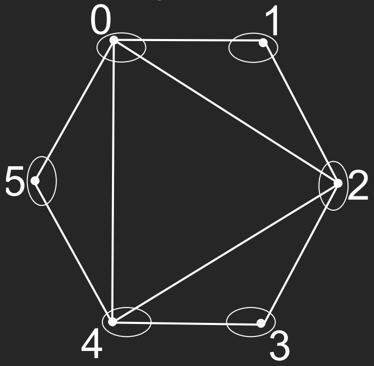
Just white triangles:



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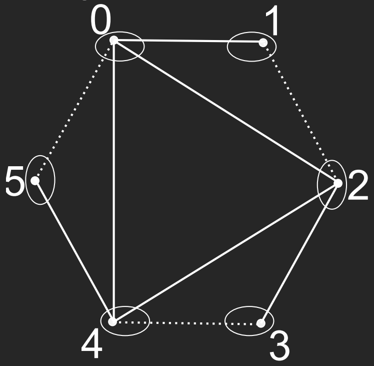
Excess edges:



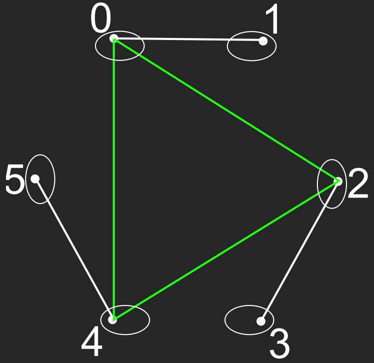
Example of the transformation

- Add new blocks to get an optimal $3 - GDPD$ of type 1^6

$V = \mathbb{Z}_6$
$\mathcal{G} = \{\{i\} : i \in \mathbb{Z}_6\}$
$\mathcal{B}$:

$\{1, \quad 2, \quad 5\}$
$\{0, \quad 3, \quad 5\}$
$\{1, \quad 3, \quad 4\}$
$\{0, \quad 2, \quad 4\}$

New block and leave edges:



Construct GDPDs

- easiest case: block size $k = 3$

Construct GDPDs

- easiest case: block size $k = 3$
- any number of groups u
- any group size g

Size

Corollary

Let g and $u \geq 3$ be positive integers. Then, $D(3, g^u) \leq U(3, g^u)$, where

$$U(3, g^u) = \left\lfloor \frac{gu}{3} \left\lfloor \frac{g(u-1)}{2} \right\rfloor \right\rfloor - \delta,$$

such that

$$\delta = \begin{cases} 1, & u \equiv 2 \pmod 6 \text{ and } g \equiv 2, 4 \pmod 6, \\ 1, & u \equiv 5 \pmod 6 \text{ and } g \not\equiv 0 \pmod 3, \\ 0, & \text{otherwise.} \end{cases}$$

Corollary (Heinrich and Yin (1999))

$$C(3, g^u) = \left\lceil \frac{gu}{3} \left\lceil \frac{g(u-2)}{2} \right\rceil \right\rceil.$$

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.... finally I have a solution!

Number of edges in excess and leave graphs

Excess graph of optimal $3-GDCD$s (mod 6):

$u\backslash g$	0	1	2	3	4	5
0	0	$\frac{gu}{2}$	0	$\frac{gu}{2}$	0	$\frac{gu}{2}$
1	0	0	0	0	0	0
2	0	$\frac{gu}{2}+1$	2	$\frac{gu}{2}$	2	$\frac{gu}{2}$
3	0	0	0	0	0	0
4	0	$\frac{gu}{2}+1$	0	$\frac{gu}{2}$	0	$\frac{gu}{2}+2$
5	0	2	2	0	2	2

Leave graph of optimal $3-GDPD$s (mod 6):

$u\backslash g$	0	1	2	3	4	5
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1	0	0	0	0	0	0
2	0	$\frac{gu}{2}$	4	$\frac{gu}{2}$	4	$\frac{gu}{2}+2$
3	0	0	0	0	0	0
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2	0	$\frac{gu}{2}$	4	$\frac{gu}{2}$	4	$\frac{gu}{2}+2$
3	0	0	0	0	0	0
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3	0	0	0	0	0	0
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1	0	0	0	0	0	0
2	0	$\frac{gu}{2}$	4	$\frac{gu}{2}$	4	$\frac{gu}{2}+2$
3	0	0	0	0	0	0
4	0	$\frac{gu}{2}+1$	0	$\frac{gu}{2}$	0	$\frac{gu}{2}+2$
5	0	4	4	0	4	4

Construction methods

Construction methods which use “transformable” ingredients:

- Wilson’s construction (for groups)
- Double group divisible designs (for group sizes)

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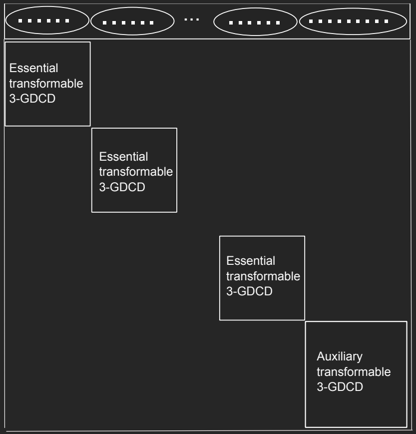
- Wilson's construction (for groups)
- Double group divisible designs (for group sizes)

Two types of ingredients:

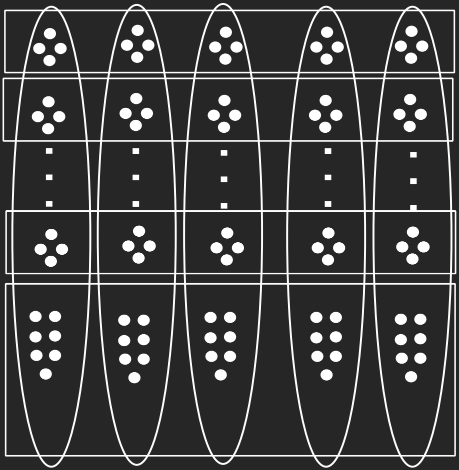
- essential (regular excess/leave graphs)
- auxiliary (excess/leave graphs have 2 vertices of higher degree)

Wilson construction

Impose structure of a GDD on groups and replace all blocks and groups by optimal transformable $3-GDCD$s



Double group divisible construction



Results for Transformable $3-GDCD$s to $3-GDPD$s

Theorem

There exists an optimal $3-GDCD$ of type g^u which transforms into an optimal $3-GDPD$ of type g^u with $U(3,g^u)$ blocks if $u \geq 3$ and one of the following holds:

- $g \geq 1$ *and* $u \equiv 0,1,3,5 \pmod 6$,
- $g \equiv 0,2,4 \pmod 6$ *and* $u \equiv 2,4 \pmod 6$,
- $g \equiv 3 \pmod 6$ *and* $u \equiv 2,4 \pmod 6$,
- $g \equiv 1 \pmod 6$ *and* $u \equiv 2,4 \pmod 6$, *except possibly when* $g = 7$ *or* $u \in \{10,14,16,20\}$,
- $g \equiv 5 \pmod 6$ *and* $u \equiv 4 \pmod 6$, *except possibly when* $g = 11$ *or* $u \in \{10,14,16,20\}$.

What is left to be done?

Excess graph of optimal $3-GDCD$s:

$u\backslash g$	0	1	2	3	4	5
0	0	$\frac{gu}{2}$	0	$\frac{gu}{2}$	0	$\frac{gu}{2}$
1	0	0	0	0	0	0
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3	0	0	0	0	0	0
4	0	$\frac{gu}{2}+1$	0	$\frac{gu}{2}$	0	$\frac{gu}{2}+2$
5	0	2	2	0	2	2

Leave graph of optimal $3-GDPD$s:

$u\backslash g$	0	1	2	3	4	5
0	0	$\frac{gu}{2}$	0	$\frac{gu}{2}$	0	$\frac{gu}{2}$
1	0	0	0	0	0	0
2	0	$\frac{gu}{2}$	4	$\frac{gu}{2}$	4	$\frac{gu}{2}+2$
3	0	0	0	0	0	0
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5	0	4	4	0	4	4

What is left to be done?

- Missing ingredient: transformable $3-GDCD$ of type 5^8

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- Missing ingredient: transformable $3 - GDCD$ of type 5^8
- Left over values: transformable $3 - GDCD$s of type:
  - $g \in \{7, 11\}$ and $u \in \{4, 8\}$
  - $g \in \{5, 7, 11\}$ an $u \in \{10, 14, 16, 20\}$

Results for optimal $3-GDPDs$

Theorem

Let g and $u \geq 3$ be positive integers. Then,

$$D(3, g^u) = \left\lfloor \frac{u}{3} \left\lfloor \frac{(u-1)}{2} \right\rfloor \right\rfloor - \delta,$$

where

$$\delta = \begin{cases} 1, & g \equiv 2, 4 \pmod 6 \text{ and } u \equiv 2 \pmod 6, \\ 1, & g \not\equiv 0 \pmod 3 \text{ and } u \equiv 5 \pmod 6, \\ 0, & \text{otherwise}, \end{cases}$$

except possibly when $g \in \{7, 11\}$ and $u \in \{10, 14, 16, 20\}$.

Thank you!