Dimension and designs

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University of Victoria | Mathematics and Statistics

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Linear spaces

An *incidence structure* is a triple $(X, \mathcal{L}, \iota)$, where

- X is a set of *points*,
- $\mathcal{L}$ is a set of *lines*, and
- $\iota \subset X \times \mathcal{L}$ is a set of *flags*.

We say $x \in X$ is *incident with* $L \in \mathcal{L}$ if and only if $(x, L) \in \iota$.

A *linear space* is an incidence structure $(X, \mathcal{L}, \iota)$ with the property that any two distinct points are both incident with (on) exactly one line and every line is on at least two points.

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Pairwise balanced designs

Linear spaces appear in another context as *pairwise balanced designs*, or *PBDs*. Specifically, if v is a positive integer and K is a set of positive integers, a PBD(v, K) consists of a v-set X, together with a set $\mathcal{B}$ of *blocks*, where

- for each $B \in \mathcal{B}$, we have $B \subset X$ with $|B| \in K$; and
- every two distinct elements of X appear together in exactly one block.

Usually, $K \subset \{3, 4, 5, \dots\}$.

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Dimension

The subspace (subdesign) *generated by* $Y \subset X$ is the unique minimal subspace containing Y. This is the intersection of all subspaces containing Y.

The *dimension* of a linear space is the maximum integer d such that any set of d points generates a proper subspace.

Since the subspace generated by any two points is simply a line, every linear space $(X, \mathcal{L}, \iota)$ with $|\mathcal{L}| > 1$ has dimension at least two.

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Steiner spaces

A PBD(v, $\{3\}$) is also known as a *Steiner triple system.* It is well-known that these exist if and only if
$v \equiv 1$ or $3 \pmod 6$.

A *Steiner space* is a Steiner triple system of dimension at least 3.

Theorem. (Teirlinck) Suppose $v \equiv 1$ or $3 \pmod 6$ and $v \notin \{51, 67, 69, 145\}$. There exists a Steiner space on v points if and only if $v = 15, 27, 31, 39$, or $v \geq 45$.

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Affine space

Let q be a prime power and $\mathbb{F}_q$ the finite field of order q.

Consider the vector space $X = \mathbb{F}_q^d$, equipped with all affine translates of its subspaces. This is the *affine space* $\mathrm{AG}_d(q)$.

If we let $\mathcal{B}$ be the set of all lines, then $(X, \mathcal{B})$ forms a linear space of dimension d.

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Main result

Theorem
For $k \in \mathbb{Z}_{\geq 2}$ and $d \in \mathbb{Z}_+$, there exists a PBD$(v, \{k\})$ of dimension at least d for all sufficiently large v satisfying

$$\begin{aligned} v - 1 &\equiv 0 \pmod{k-1}, \\ v(v-1) &\equiv 0 \pmod{k(k-1)}, \end{aligned}$$

That is, we can demand dimension $\geq d$ in Wilson's theorem.

Main result

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Proof sketch

Work from the affine space $\mathrm{AG}_d(q)$.

Step 1. For any positive integer d and any prime power q, there exists a q-GDD of *strong dimension* $\geq d$ and type $n^{(q^d)} := [\underbrace{n, \ldots, n}_{q^d}]$ for all sufficiently large integers n.

Proof sketch

Now, break up blocks of size q by replacing them with PBD$(q, \{r\})$ when possible. We can then truncate points from the last group, dropping some block sizes by one.

Step 2. For any positive integers d and r with $r \geq 3$, there exists an $\{r-1, r\}$-GDD of strong dimension $\geq d$ and type $n^{(q^d-1)}x^1$ for all large prime powers $q \equiv 1 \pmod{r(r-1)}$, all sufficiently large integers n, and for any positive integer $x \leq n$.

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Proof sketch

Step 3. For any positive integers d and k with $k \geq 2$, there exists a k-GDD of dimension $\geq d$ and type $[n(k-1)]^{(q^d-1)}[x(k-1)]^1$ for large prime powers $q \equiv 1 \pmod{k(k-1)}$, all sufficiently large integers n, and for any positive integer $x \leq n$.

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Proof sketch

Add a point and fill with PBDs having appropriate replication numbers.

Step 4. For any positive integers d and k with $k \geq 2$, there exists a PBD of blocksize k, dimension $\geq d$, and replication number $n(q^d - 1) + x$ for infinitely many prime powers $q \equiv 1 \pmod{k(k-1)}$, all sufficiently large integers n with $k \mid n$, and for any integer x with $r_0(k) \leq x \leq n$ and $x(x-1) \equiv 0 \pmod{k}$.

Proof sketch

Vary x and show that intervals of constructible replication numbers overlap for large n.

Step 5. Any sufficiently large y with $y(y-1) \equiv 0 \pmod{k}$ can be expressed as $y = n(q^d - 1) + x$, $n \geq n_0(q)$, $r_0(k) \leq x \leq n$.

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Suppose we slightly strengthen the dimension 3 requirement by (universally) bounding the three-point generated subspaces as v grows.

For example, it is known that, for all integers v, there exist $\mathrm{PBD}(v, \{3, 4, 5\})$ such that any three points generate a subspace of size < 1000.

Consider now one-factorizations of the complete bipartite graphs $K_{n,n}$ (a.k.a. n-edge-colourings of $K_{n,n}$/ a.k.a latin squares).

Align idempotent sub-latin squares of orders $3, 4, 5$ on the blocks of a $\mathrm{PBD}(n, \{3, 4, 5\})$ with bounded three-point generated subspaces. This construction universally bounds the longest bi-colored cycle!

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