Realization polytopes for the degree sequence of a graph

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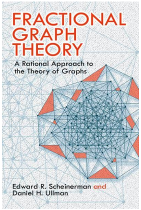
A fractional excursion



It is possible to go to a graph theory conference and to ask oneself, at the end of every talk, What is the fractional analogue? What is the right definition?
— Scheinerman and Ullman, *Fractional Graph Theory*

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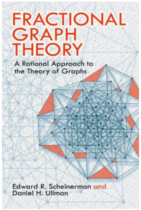

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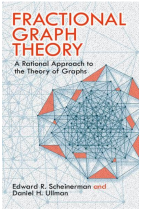

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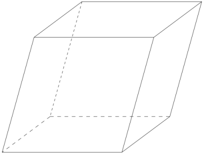
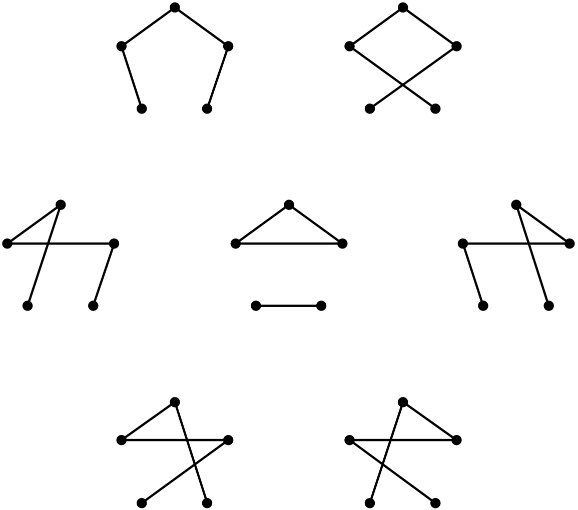
U.N. Peled and M.K. Srinivasan. The polytope of degree sequences. Linear Algebra and its Applications, 114/115:349–377 (1989).



Keeping track of realizations

$(2, 2, 2, 1, 1)$



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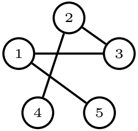
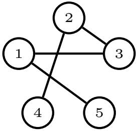


(0,	1,	0,	1,	1,	1,	0,	0,	0,	0) $\in \mathbb{R}^{10}$
12	13	14	15	23	24	25	34	35	45

Keeping track of realizations



$$\underset{12}{(0,}\quad \underset{13}{1,}\quad \underset{14}{0,}\quad \underset{15}{1,}\quad \underset{23}{1,}\quad \underset{24}{1,}\quad \underset{25}{0,}\quad \underset{34}{0,}\quad \underset{35}{0,}\quad \underset{45}{0)} \in \mathbb{R}^{10}$$

Simple graph realizations of $(2, 2, 2, 1, 1)$ must satisfy

$$x_{12} + x_{13} + x_{14} + x_{15} = 2$$
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$$x_{13} + x_{23} + x_{34} + x_{35} = 2$$
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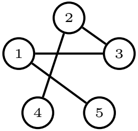
$$x_{ij} \in \{0, 1\}$$

Keeping track of realizations



$$\begin{array}{cccccccccc} (0, & 1, & 0, & 1, & 1, & 1, & 0, & 0, & 0, & 0) \in \mathbb{R}^{10} \\ 12 & 13 & 14 & 15 & 23 & 24 & 25 & 34 & 35 & 45 \end{array}$$

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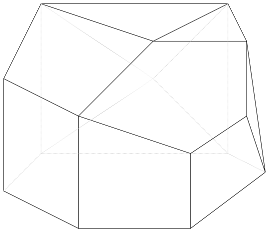
$$0 \leq x_{ij} \leq 1$$

A polytope

Given d, let $S(d)$ be the polytope in $\mathbb{R}^{\binom{n}{2}}$ defined by

Degree conditions

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} x_{ij} = d_i \qquad \text{for } 1 \leq i \leq n$$



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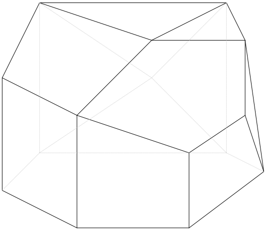

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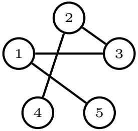
What are the vertices?

Two polytopes



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$$
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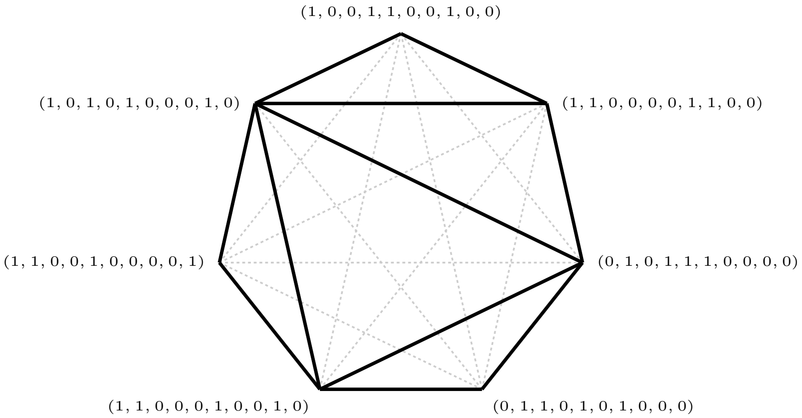
\end{gathered}
$$

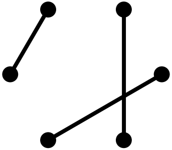
$S(d)$: bounded by hyperplanes $\qquad$ $R(d)$: convex hull

$S(2, 2, 2, 1, 1)$



Here, $R(d) = S(d)$, i.e., vertices correspond exactly to simple graph realizations.

$S(1, 1, 1, 1, 1, 1)$



$$\sum_{\substack{j=1 \\ j \neq i}}^{n} x_{ij} = 1, \qquad 0 \leq x_{ij} \leq 1$$

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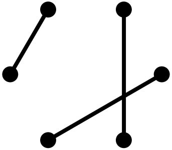

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$(1,1,1,1,1,1)$ has **15** realizations, but $S(1,1,1,1,1,1)$ has **25** vertices, so $R(d) \neq S(d)$.

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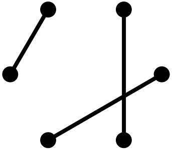

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A non-integral vertex: $(0,0,0,1/2,1/2,1/2,1/2,0,0,1/2,0,0,0,0,1/2)$

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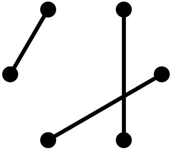

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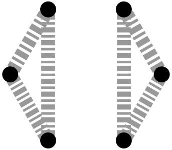
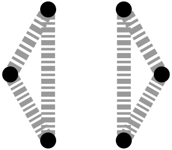


Each fractional edge has value $1/2$.

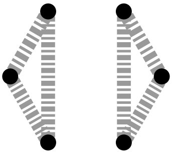
Questions



$(0, 0, 0, 1/2, 1/2, 1/2, 1/2, 0, 0, 1/2, 0, 0, 0, 0, 1/2)$

- What determines whether the vector of a fractional realization is a vertex of $S(d)$?

Questions



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- What determines whether the vector of a fractional realization is a vertex of $S(d)$?

- Which degree sequences, like $(2, 2, 2, 1, 1)$, satisfy $S(d) = R(d)$, i.e., the vertices of $S(d)$ correspond exactly to simple graph realizations of d?

Vertices of $S(d)$?

$$(1,0,1,0,1,0,0,0,1,0) \qquad (2/3,1,0,1/3,2/3,2/3,0,0,1/3,1/3)$$
$$(0,0,0,1/2,1/2,1/2,1/2,0,0,1/2,0,0,0,0,1/2)$$

Vertices of $S(d)$?

$(1, 0, 1, 0, 1, 0, 0, 0, 1, 0)$ $(2/3, 1, 0, 1/3, 2/3, 2/3, 0, 0, 1/3, 1/3)$

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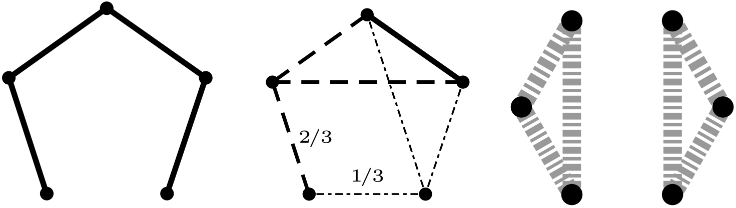

Vertices of $S(d)$?

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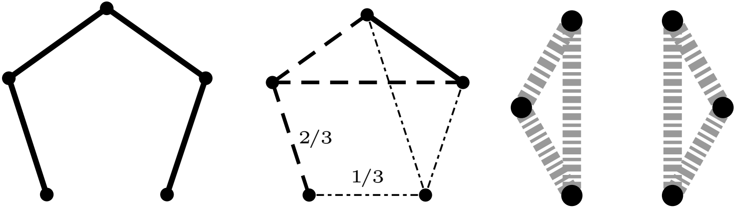

Theorem

Given a graphic sequence d, a point in $S(d)$ is a vertex of $S(d)$ if and only if the non-integral edges in the corresponding realization form a disjoint union of odd cycles.
When this is the case, there are an even number of these cycles, and each non-integral edge has value $1/2$.

When can/can't this happen?

$$(1, 1, 1, 1, 1, 1) \qquad \text{vs} \qquad (2, 2, 2, 1, 1)$$

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For d **not** admitting disjoint odd cycles of $1/2$-edges,

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For *d* **not** admitting disjoint odd cycles of $1/2$-edges,

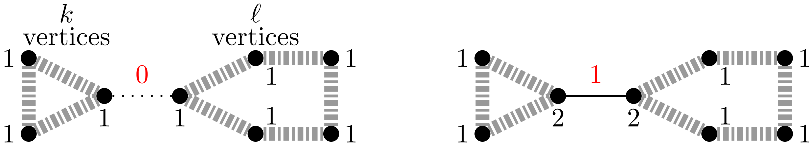
- $S(d) = R(d)$, i.e., vertices of $S(d)$ correspond exactly to simple graph realizations of *d*.
- In **every** realization, each edge is either **100% there** or **not there** — no halfways about it.

We call such *d* **forceful sequences**, and we call their realizations **forceful graphs**.

How do we recognize forceful sequences/graphs?

What happens in nonforceful graphs

For odd k, ℓ:



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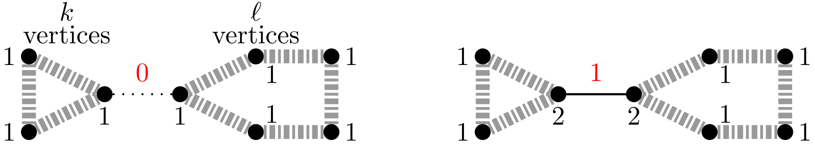
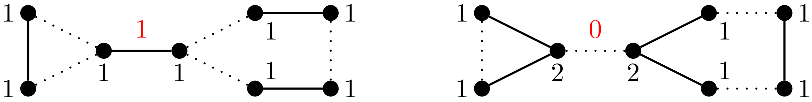
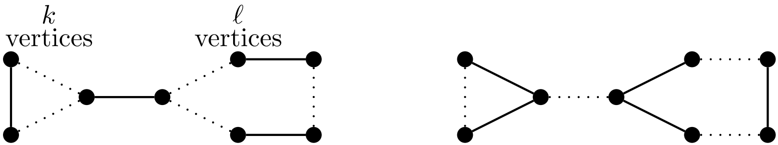


A switch...



We get another vertex of $S(d)$ from a (fractional) realization with fewer $1/2$-edge cycles...

(k, ℓ)-blossoms

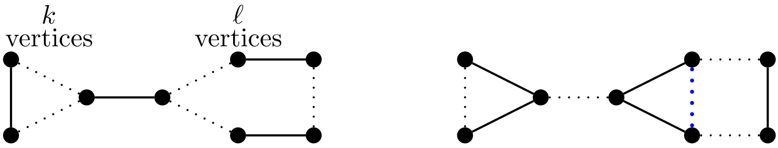


Theorem

For a graphic sequence d, the following are equivalent:

- *d is a forceful sequence;*
- *No realization of d contains a (k, ℓ)-blossom;*
- *No realization of d contains a $(3, 3)$-blossom.*

(k,ℓ)-blossoms



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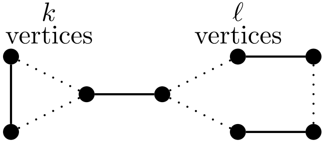

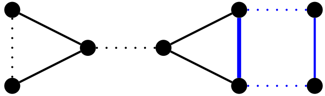

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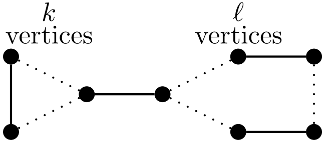

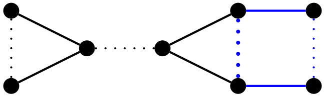

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Forbidden subgraphs

d is forceful **iff** no realization of d contains a $(3, 3)$-blossom.

The forceful graphs form a hereditary class. What are the forbidden subgraphs?

Forbidden subgraphs

d is forceful **iff** no realization of d contains a $(3,3)$-blossom.

The forceful graphs form a hereditary class. What are the forbidden subgraphs?

We start by generating all 6-vertex graphs we can build up from a $(3,3)$-blossom...

...as well as all graphs having the same degree sequence as one of these.

Denote the set of all these (forbidden sub)graphs as $\mathcal{B}$.

Forbidden "subsequences"

d is forceful $\implies$ every realization of d is $\mathcal{B}$-free

The class $\mathcal{B}$ contains 70 graphs—all realizations of the following degree sequences:

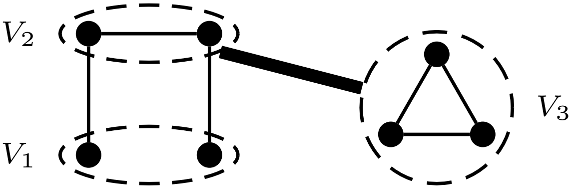
$$\begin{array}{llll}
(1,1,1,1,1,1), & (3,3,2,2,1,1), & (4,2,2,2,1,1), & (4,4,3,3,2,2),\\
(2,2,1,1,1,1), & (3,3,2,2,2,2), & (4,3,2,2,2,1), & (4,4,3,3,3,1),\\
(2,2,2,2,1,1), & (3,3,3,2,2,1), & (4,3,3,2,2,2), & (4,4,3,3,3,3),\\
(2,2,2,2,2,2), & (3,3,3,3,1,1), & (4,3,3,3,2,1), & (4,4,4,3,3,2),\\
(3,2,2,1,1,1), & (3,3,3,3,2,2), & (4,3,3,3,3,2), & (4,4,4,4,3,3),\\
(3,2,2,2,2,1), & (3,3,3,3,3,3), & (4,4,2,2,2,2), & (4,4,4,4,4,4).
\end{array}$$

What structure does this impose on forceful graphs?

Decomposable graphs

A graph is ***decomposable*** if its vertex set can be partitioned into V_1, V_2, V_3 with $V_1 \cup V_2, V_3 \neq \emptyset$ such that

- V_1 is an independent set,
- V_2 is a clique, and
- every vertex of V_3 is adjacent to all vertices of V_2 and to no vertices of V_1.



(See work by Blázsik et al. '93, Tyshkevich '00, and others)

Forceful graph structure

Theorem

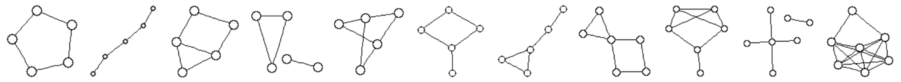
Let d be a graphic list. The following are equivalent:

(1) *d is forceful;*

(2) *Every realization of d is $\mathcal{B}$-free;*

(3) *For every realization G of d, either G is split or G is decomposable with $G[V_3]$ equal to one of the following:*

C_5, P_5, house, $K_3 + K_2$, $K_{2,3}$, 4-pan, co-4-pan, U_1, $\overline{U_1}$, $K_2 + K_{1,m}$, or $(K_m + K_1) \vee 2K_1$ for $m \geq 1$;



(4) **Any** *realization of d satisfies the property in either (2) or (3).*

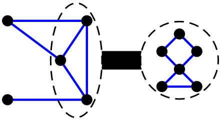
Threshold graphs, split, pseudo-split $\subset$ forceful $\subsetneq$ perfect

Forceful sequences, on their own terms

Theorem

For a graphic sequence d, the following are equivalent:

- *d is a forceful sequence;*
- *When d is "pruned" according to any Erdős–Gallai equalities, the resulting sequence is empty (i.e., d is a split sequence) or one of* $\{(2,2,2,1,1),(2,2,2,2,2),(3,2,2,2,1),(3,3,2,2,2),$ $(3,3,3,3,3,1),(4,2,2,2,2,2),(m,1^{(m+2)}),((m+1)^{(m+2)},2)\}$.



(9,9,9,7,5,5,5,5,5,2,1)

$$\sum_{i=1}^{3} d_i = 3(3-1) + \sum_{i>3} \min\{3, d_i\}$$

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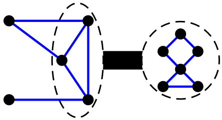
- *When d is "pruned" according to any Erdős–Gallai equalities, the resulting sequence is empty (i.e., d is a split sequence) or one of* $\{(2,2,2,1,1),(2,2,2,2,2),(3,2,2,2,1),(3,3,2,2,2),$ $(3,3,3,3,3,1),(4,2,2,2,2,2),(m,1^{(m+2)}),((m+1)^{(m+2)},2)\}$.



$(9,9,9,7,5,5,5,5,5,2,1) \rightarrow (4,2,2,2,2,2)$

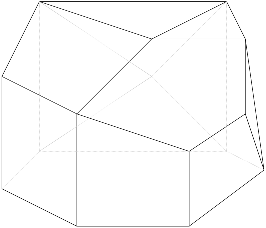
$$\sum_{i=1}^{3} d_i = 3(3-1) + \sum_{i>3} \min\{3, d_i\}$$

Where we've been

Given d, let $S(d)$ be the polytope in $\mathbb{R}^{\binom{n}{2}}$ defined by

Degree conditions

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} x_{ij} = d_i \qquad \text{for } 1 \leq i \leq n$$



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$$0 \leq x_{ij} \leq 1 \qquad \text{for } 1 \leq i, j \leq n$$

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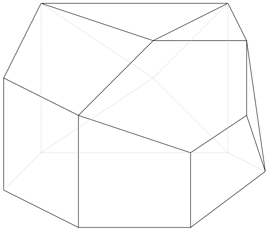

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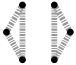
What are the vertices? When is $S(d) = R(d)$?

Current and future work

- What are the "extra" inequalities (besides the degree conditions and hypercube bounds) that define $R(d)$? 

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- Identifying realizations/isomorphism types via objective functions?

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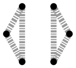
- What are the "extra" inequalities (besides the degree conditions and hypercube bounds) that define $R(d)$? 
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Thank you!