Unlabeled Motzkin numbers

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Catalan numbers can be defined by the explicit formula:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}$$

and the ordinary generating function:

$$\mathcal{C}(x) = \sum_{n=0}^{\infty} C_n \cdot x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

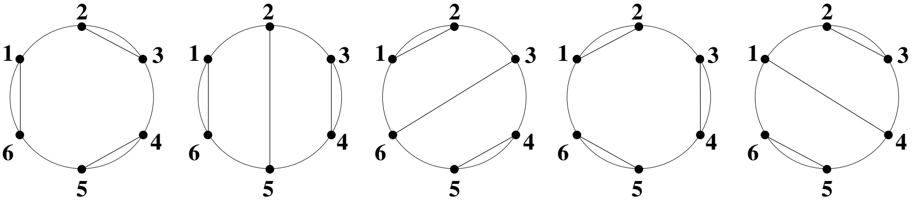
Catalan numbers for $n = 0, 1, \dots$ form the sequence (A000108 in OEIS):

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \dots$$

We are particularly interested in the combinatorial interpretation of C_n as the number of expressions containing n properly embedded pairs of parentheses. For $n = 3$, these expressions are:

(( ) ( ))	((( )))	( ) (( ))	( ) ( ) ( )	(( )) ( )
1 2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6

which can be further interpreted as the number of configurations of n noncrossing chords connecting $2n$ labeled points on a circle:



Motzkin numbers

Catalan number C_n represents the number of configurations of n noncrossing chords connecting $2n$ labeled points on a circle.

Motzkin number M_n represents the number of configurations of (any number of) noncrossing chords connecting n labeled points on a circle.

We can easily expressed Motzkin numbers in terms of Catalan numbers:

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k.$$

The generating functions of Motzkin numbers is:

$$\mathcal{M}(x) = \sum_{n=0}^{\infty} M_n \cdot x^n = \frac{1}{1-x} \cdot \mathcal{C}\left(\frac{x^2}{(1-x)^2}\right) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}.$$

Unlabeled Motzkin numbers

Consider a circle with n equally spaced points, which we will call *vertices*. A set of noncrossing chords connecting vertices is called a *chord configuration*.

We define two types of *unlabeled Motzkin numbers* counting the number of chord configurations on *unlabeled* vertices. Namely, we define *cyclic* and *dihedral* Motzkin numbers counting the number of chord configurations up to the action of cyclic and dihedral groups, respectively.

Cyclic and dihedral Motzkin numbers

Cyclic Motzkin number M_n^C represents the number of chord configurations on n vertices under the action of the cyclic group (of *rotations*) C_n. Burnside lemma allows us to give the following expression for M_n^C.

$$M_n^C = \frac{1}{n} \sum_{c \in C_n} H^c, \qquad (1)$$

where H^c is the number of chord configurations invariant w.r.t. c.

Similarly, *dihedral Motzkin number* M_n^D represents the number of chord configurations on n vertices under the action of the dihedral group D_n. Viewing elements of D_n as n rotations, forming the cyclic subgroup C_n, and n *reflections*, forming a set R_n, we compute M_n^D as follows:

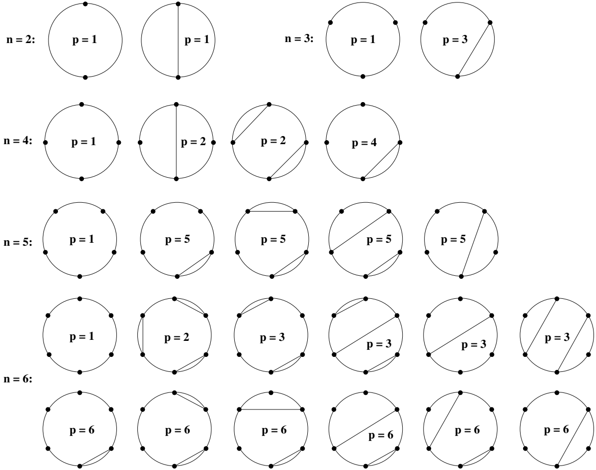
$$M_n^D = \frac{1}{2n} \left(\sum_{c \in C_n} H^c + \sum_{r \in R_n} H^r \right) = \frac{1}{2} M_n^C + \frac{1}{2n} \sum_{r \in R_n} H^r. \qquad (2)$$

We define the *period* of a chord configuration S as the smallest positive integer p such that S is invariant w.r.t. rotation of the circle by the angle $p \cdot \frac{2\pi}{n}$.
Clearly, the period of any chord configuration on n vertices divides n.

A chord configuration on n vertices is called *special* if it contains a chord connecting two diametrically opposite vertices.

- Special configurations exist only for even n.
- Period of a special configuration can be only n or $n/2$.
- The number of special configurations of period $n/2$ equals $M_{n/2-1}$.
- The number of special configurations of period n is $\binom{M_{n/2-1}}{2}$.

Below we list of all configurations of chords connecting n ($2 \le n \le 6$) vertices and specify their periods p.



Nonspecial chord configurations

A chord c in a nonspecial chord configuration partition the vertices other than the endpoints of c into two subsets formed by vertices laying at the same side of c.
We define the *span* of c as the smaller of these subsets together with the endpoints of c.

For a configuration of period $m < n$, the span of each chord does not exceed m.

A chord is called *maximal* if its endpoints do not reside within the span of any other chord. It is easy to see that all chords in a chord configuration reside within the spans of the maximal chords.

Let $b(n, m)$ be the number of nonspecial configurations on n vertices, whose period equals m. Clearly, $b(n, m)$ can be non-zero only if m divides n.

Rotation of a chord configuration of period m by the angle $m \cdot \frac{2\pi}{n}$ translates maximal chords into maximal chords. Therefore, the number of maximal chords in such configuration must be a multiple of ${}^{n}/_{m}$.

For $m \mid n$, define $b(n, m, k)$ as the number of chord configurations on n vertices of period m with $k \cdot {}^{n}/_{m}$ maximal chords.
Similarly, let $c(n, m, k)$ be the number of such configurations with a labeled maximal chord.

Clearly,

$$b(n, m) = \sum_{k \geq 0} b(n, m, k).$$

Formula for $b(n, m, k)$

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Theorem

For $m \mid n$, $m < n$, *and* $k \geq 1$, $c(n, m, k)$ *equals the coefficient of* $x^m y^k$ *in*

$$\left(1 - y\mathcal{M}(x)\frac{x^2}{1-x}\right)^{-1} = \left(1 - y\frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2(1-x)}\right)^{-1}.$$

Lemma

For $m \mid n$ *and* $k \geq 1$,

$$c(n, m, k) = \sum_{d \mid (m,k)} b(n, {}^m\!/_d, {}^k\!/_d) \cdot {}^k\!/_d.$$

Lemma

For $m \mid n$, *we have* $b(n, m, 0) = [m = 1]$ *and for* $k \geq 1$,

$$b(n, m, k) = \frac{1}{k} \sum_{d \mid (m,k)} \mu(d) \cdot c(n, {}^m\!/_d, {}^k\!/_d),$$

where $\mu(\cdot)$ *is Möbius function.*

Proof.

Consider an arbitrary chord configuration on n vertices with period dividing m and containing $k \cdot {}^{n}/_{m}$ maximal chords, one of which is labeled. Let P be set of m consecutive vertices on the circle that starts with the counterclockwise endpoint of the labeled maximal chord and goes clockwise. Then P contains the spans of k maximal chords. Let t_i $(1 \leq i \leq k)$ be the size of the span of the i-th (counting clockwise) maximal chord in P. Then the number of chord configurations within this span is M_{t_i-2}. Let z_i $(1 \leq i \leq k)$ be the number of vertices between the endpoints of i-th and $(i+1)$-th maximal chords (or the end of P for $i = k$) so that the total number of vertices is

$$t_1 + z_1 + t_2 + z_2 + \cdots + t_k + z_k = m.$$

Then $c(n, m, k)$ as the total number of chords configurations within P equals

$$\sum_{t_1+z_1+\cdots+t_k+z_k=m} M_{t_1-2} \cdot M_{t_2-2} \cdots M_{t_k-2} = [x^{m-2k}]\mathcal{M}(x)^k(1-x)^{-k} = [x^m]\left(\mathcal{M}(x)\frac{x^2}{1-x}\right)^k.$$

We multiply this by y^k and sum over $k \geq 0$ to get

$$c(n, m, k) = [x^m y^k]\left(1 - y\mathcal{M}(x)\tfrac{x^2}{1-x}\right)^{-1}.$$

Generating function for $b(n, m)$

We define the following functions:

$$\mathcal{T}(x) = -\ln\left(1 - x - x^2 \cdot \mathcal{M}(x)\right),$$

$$\mathcal{B}(x) = \sum_{q=1}^{\infty} \frac{\mu(q)}{q} \cdot \mathcal{T}(x^q) = \ln \prod_{q=1}^{\infty} \left(1 - x^q - x^{2q} \cdot \mathcal{M}(x^q)\right)^{-\mu(q)/q},$$

$$\mathcal{F}(x) = \sum_{q=1}^{\infty} \frac{\varphi(q)}{q} \cdot \mathcal{T}(x^q) = \ln \prod_{q=1}^{\infty} \left(1 - x^q - x^{2q} \cdot \mathcal{M}(x^q)\right)^{-\varphi(q)/q},$$

where $\varphi(\cdot)$ is Euler totient function.

Lemma

For positive integers $m \mid n$ with $m < n$,

$$b(n, m) = [x^m]\ \mathcal{B}(x).$$

Configurations of fixed period

Let $b'(n,m)$ be the number of chord configurations (both special and nonspecial) whose period equals m. Clearly, $b'(n,m)$ can be non-zero only if m divides n, $M_n = \sum_{m|n} b'(n,m) \cdot m$, and $M_n^C = \sum_{m|n} b'(n,m)$.

Lemma

For $m|n$, we have $b'(n,m) = b(n,m)$ if $m < n/2$.
Furthermore, $b'(n, n/2) = b(n, n/2) + M_{n/2-1}$ if n is even, and

$$b'(n,n) = \frac{1}{n}\left(M_n - \sum_{\substack{m|n \\ m<n}} b'(n,m) \cdot m\right).$$

Theorem

The generating function for the number of asymmetric chord configurations $b'(n,n)$ is

$$\sum_{n=0}^{\infty} b'(n,n)\cdot x^n = \frac{1-(1-x)\cdot\mathcal{M}(x)-x^2\cdot\mathcal{M}(x^2)}{2} + \ln(\mathcal{M}(x)) + \mathcal{B}(x) - \mathcal{T}(x)$$

$$= \frac{1-(1-x)\cdot\mathcal{M}(x)-x^2\cdot\mathcal{M}(x^2)}{2} + \ln(\mathcal{M}(x)) + \ln\prod_{q\geq 2}\left(1-x^q-x^{2q}\cdot\mathcal{M}(x^q)\right)^{-\mu(q)/q}.$$

Theorem

$$\sum_{n=0}^{\infty} M_n^C\cdot x^n = \frac{1-(1-x)\cdot\mathcal{M}(x)+x^2\cdot\mathcal{M}(x^2)}{2} + \ln(\mathcal{M}(x)) + \mathcal{F}(x) - \mathcal{T}(x)$$

$$= \frac{1-(1-x)\cdot\mathcal{M}(x)+x^2\cdot\mathcal{M}(x^2)}{2} + \ln(\mathcal{M}(x)) + \ln\prod_{q\geq 2}\left(1-x^q-x^{2q}\cdot\mathcal{M}(x^q)\right)^{-\varphi(q)/q}.$$

Lemma

For an even $n = 2m$ and a fixed reflection $r \in R_n$ about a diameter connecting centers of two arcs of the circle, we have

$$H^r = [x^m]\frac{\mathcal{M}(x)}{1 - x \cdot \mathcal{M}(x)}.$$

Lemma

For an even $n = 2m$ and a fixed reflection $r \in R_n$ about a diameter connecting two of the vertices, we have

$$H^r = [x^m]\ \frac{1}{1 - x \cdot \mathcal{M}(x)} + M_{m-1} = [x^m]\ \left(\frac{1}{1 - x \cdot \mathcal{M}(x)} + x \cdot \mathcal{M}(x)\right).$$

Lemma

For an odd $n = 2m + 1$ and a fixed reflection $r \in R_n$, we have

$$H^r = [x^m]\ \frac{\mathcal{M}(x)}{1 - x \cdot \mathcal{M}(x)}.$$

Lemma

For even $n = 2m$, we have

$$\sum_{r \in R_n} H^r = [x^m]\ m \cdot \left(\frac{1 + \mathcal{M}(x)}{1 - x \cdot \mathcal{M}(x)} + x \cdot \mathcal{M}(x) \right) = [x^n]\ \frac{n}{2} \left(\frac{1 + \mathcal{M}(x^2)}{1 - x^2 \cdot \mathcal{M}(x^2)} + x^2 \cdot \mathcal{M}(x^2) \right).$$

For odd $n = 2m + 1$,

$$\sum_{r \in R_n} H^r = [x^m]\ (2m+1) \frac{\mathcal{M}(x)}{1 - x \cdot \mathcal{M}(x)} = [x^n]\ n \frac{x \cdot \mathcal{M}(x^2)}{1 - x^2 \cdot \mathcal{M}(x^2)}.$$

Theorem

For any integer $n \geq 0$,

$$\frac{1}{2n} \sum_{r \in R_n} H^r = [x^n]\ \left(\frac{1 + (2x+1) \cdot \mathcal{M}(x^2)}{4(1 - x^2 \cdot \mathcal{M}(x^2))} + \frac{x^2 \cdot \mathcal{M}(x^2)}{4} \right).$$

Using formula $M_n^D = \frac{1}{2} M_n^C + \frac{1}{2n} \sum_{r \in R_n} H^r$, we deduce the generating function for dihedral Motzkin numbers:

$$\sum_{n=0}^{\infty} M_n^D \cdot x^n = \frac{1 - (1-x) \cdot \mathcal{M}(x) + 2x^2 \cdot \mathcal{M}(x^2)}{4} + \frac{\ln(\mathcal{M}(x)) + \mathcal{F}(x) - \mathcal{T}(x)}{2} + \frac{1 + (2x+1) \cdot \mathcal{M}(x^2)}{4(1 - x^2 \cdot \mathcal{M}(x^2))}$$

Unlabeled Motzkin number in OEIS

Currently there are three sequences in the Online Encyclopedia of Integer Sequences (OEIS) http://oeis.org related to unlabeled Motzkin numbers:

- A175954 Cyclic Motzkin numbers
- A175955 Number of asymmetric (w.r.t. rotations) chord configurations $b'(n, n)$
- A185100 Dihedral Motzkin numbers

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