# The Graph Isomorphism Problem on graphs with geometric representations

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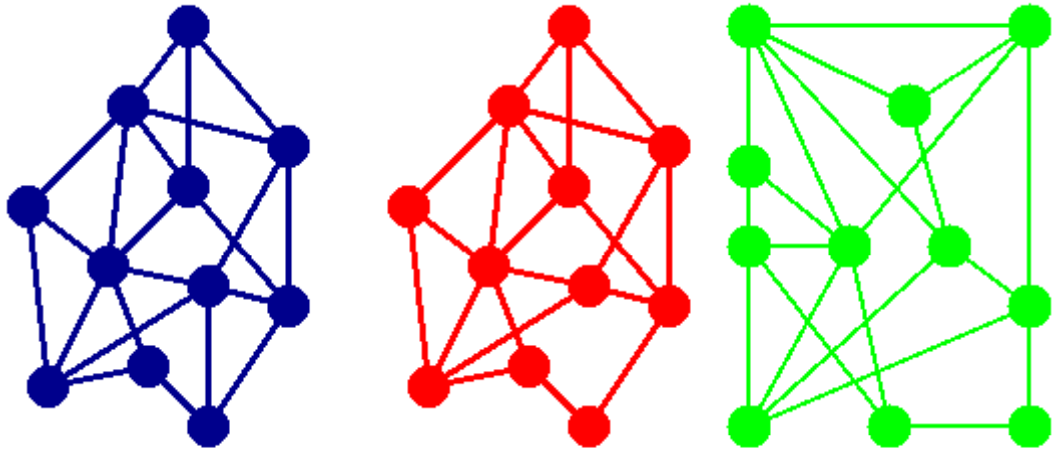
# *Graph Isomorphism Problem*

– It asks if $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ have one-to-one mapping $h$ s.t. $\{u,v\}$ in $E_1$ iff $\{h(u),h(v)\}$ in $E_2$.



– it is in $\mathcal{NP}$, but we do not know if it is $\mathcal{NP}$-complete …long standing open question since 1960s.

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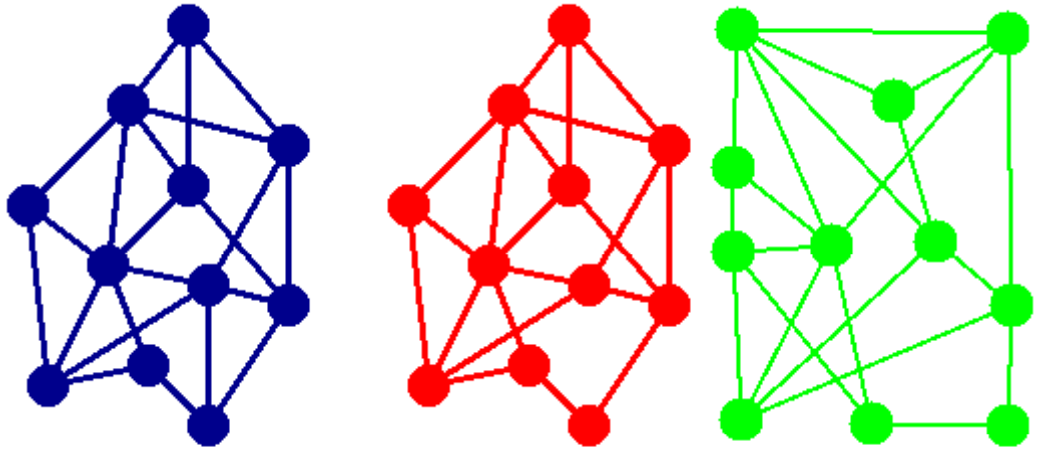

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# *Graph Isomorphism Problem*

- Research aspects from the viewpoints of theoretical computer science:
  1. Computational Complexity; how hard?

  2. Faster algorithms for general graph; exponential time algorithm

  3. Boundary between general case and polynomial solvable case; graph classes
     - ↓ as hard as general case (GI-complete)
     - ↑ polynomial time algorithms for some graph classes

# *Graph Isomorphism Problem*

- Research aspects from the viewpoints of theoretical computer science:
  1. Computational Complexity; how hard?
     - unlikely to be NP-complete

     Theorem (Boppana, Hastad, Zachos 1987; Schoning 1988)

     If GI is NP-complete, then the polynomial hierarchy collapses to its second level.

     Theorem (Toran 2004)

     GI is hard for the class DET (and hence for NL)

# *Graph Isomorphism Problem*

- Research aspects from the viewpoints of theoretical computer science:
  2. Faster algorithms for general graph; exponential time algorithm

Fast algorithm (Zemlyachenko, Babai 1981; Babai, Luks 1983)

GI can be solved in $2^{O(\sqrt{n \log n})}$

# *Graph Isomorphism Problem*

- Research aspects from the viewpoints of theoretical computer science:
  3. Boundary between general case and polynomial solvable case; graph classes
     - polynomial time solvable for the following graph classes;
       - planar graphs
         - » linear time; Hopcroft and Tarjan 1972,
         - » logspace; Datta, et al. 2009.; not correct ;-)
       - bounded genus; Filotti, Mayer 1980, Miller 1980.
       - interval graphs;
         - » linear time; Booth, Lueker 1976,
         - » $AC^2$; Klein 1996
         - » logspace; Kobler, Kuhnert, Laubner, Verbitsky 2010.

they rely on Reingold's result

bit far from implementation

# *Graph Isomorphism Problem*

- Research aspects from the viewpoints of theoretical computer science:
  3. Boundary between general case and polynomial solvable case; graph classes
     - polynomial time solvable for the following graph classes;
       - graphs with bounded eigenvalue multiplicities;
         - » Babai, Grigoriev, Mount 1982
       - bounded degree; Luks 1982
       - graphs with excluded minors; Ponomarenko 1988
       - circular arc graphs?;
         - » O(nm); Hsu 1995 … contains serious bug!!
         - » OPEN!; Curtis, et. al. 2012+ (arXiv 1203.4822, Mar. 2012)

# *Graph Isomorphism Problem*

- Research aspects from the viewpoints of theoretical computer science:
  3. Boundary between general case and polynomial solvable case; graph classes
     - GI-complete; as hard as general case for the following graph classes;
       - bipartite graphs
       - graphs of bounded degeneracy
       - graphs of bounded expansion
       - chordal graphs
       - … and some graph classes with geometric representations
         - » chordal bipartite, strongly chordal; U, Toda, Nagoya 2005
         - » grid intersection; U 2008

# *GI*-completeness for GIG

- Graph Isomorphism problem
- We say a graph class $\mathcal{C}$ is *GI*-complete if the *GI* problem for $\mathcal{C}$ is as hard as the *GI* problem for general graphs.
  - *GI* is polynomial time solvable…
    - interval graphs, interval bigraphs, etc.
  - *GI*-complete graph classes;
    - strongly chordal graphs, chordal bipartite graphs, grid intersection graphs, …

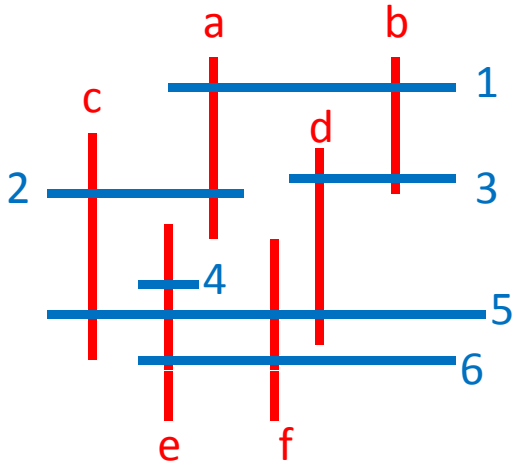
# *GI*-completeness for GIG

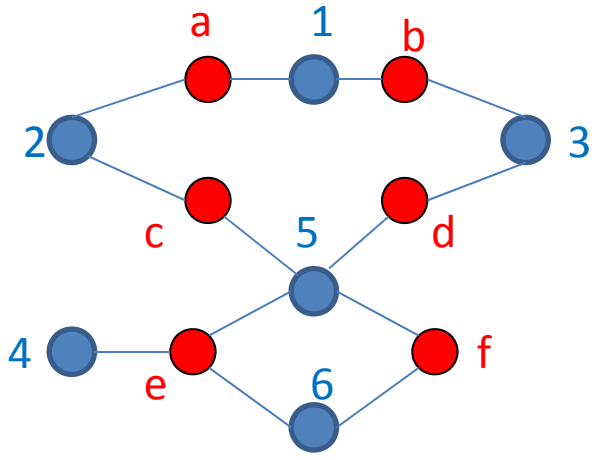
- Grid Intersection Graph $G=(X,Y,E)$ is a bipartite graph s. t.

  $X$ corresponds to vertical line segments

  $Y$ corresponds to horizontal line segments

  an edge $(x,y) \in E$ iff two lines intersect



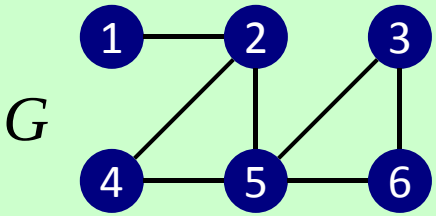



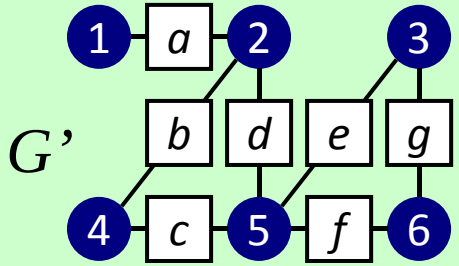
# *GI*-completeness for GIG

[Thm] Grid Intersection Graphs are *GI*-complete.

- Similar reductions can be found in
  - Babel, Ponomarenko, Tinhofer 1996.
  - Uehara, Toda, Nagoya 2004.
- Proof idea;
  - reduce any given graph $G$ to $G'$ in $\mathcal{C}$ s.t. $G_1 \sim G_2$ iff $G_1' \sim G_2'$

[Sample] Bipartite graphs are *GI*-complete



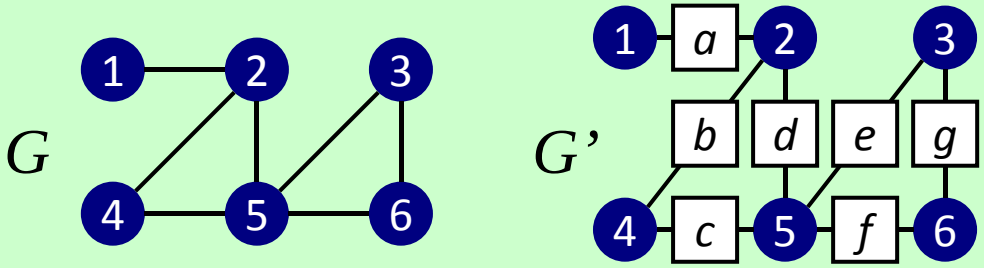




- For any graph $G$, $G'$ is bipartite.
- $G_1 \sim G_2$ iff $G_1' \sim G_2'$

# *GI*-completeness for GIG

[Thm] Grid Intersection Graphs are *GI*-complete

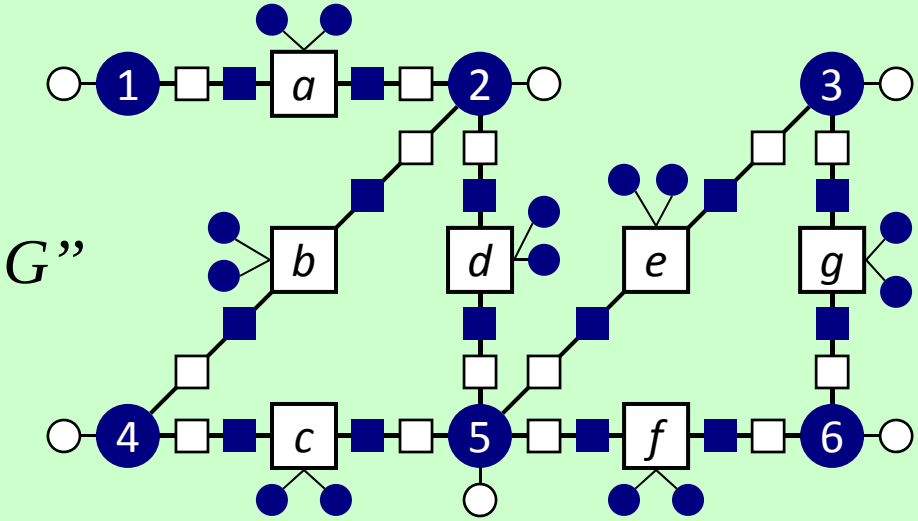
More reduction to Grid Intersection Graphs…



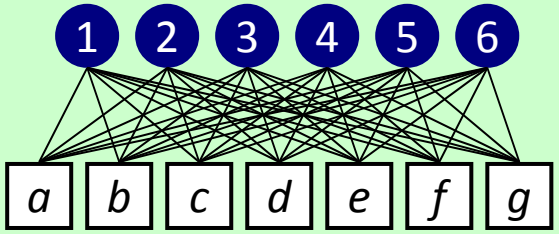




$$G_1 \sim G_2$$
$$\updownarrow$$
$$G_1' \sim G_2'$$
$$\updownarrow$$
$$G_1'' \sim G_2''$$



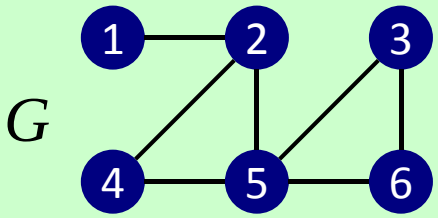

with



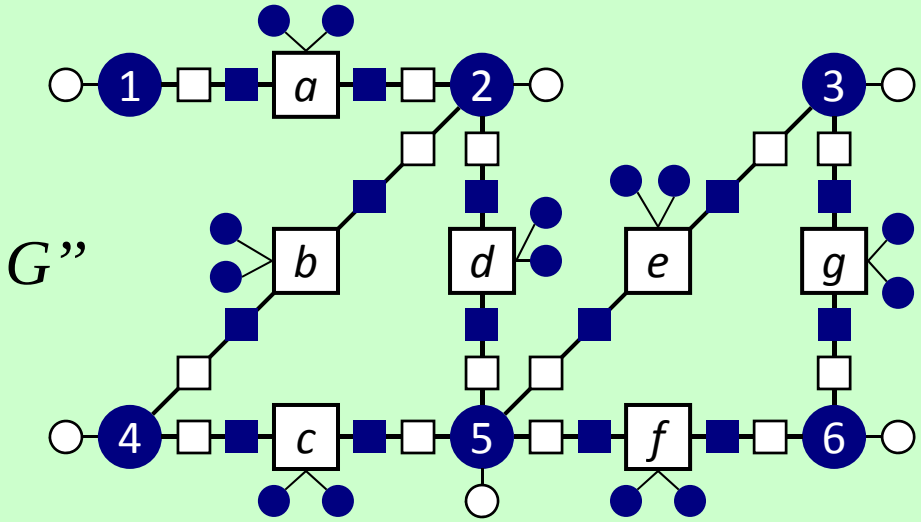
# *GI*-completeness for GIG

[Thm] Grid Intersection Graphs are *GI*-complete

More reduction to Grid Intersection Graphs…



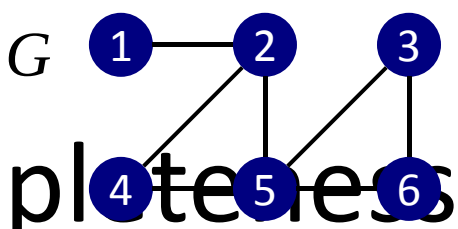

We first observe that *G* can be constructed from *G''* without labels.

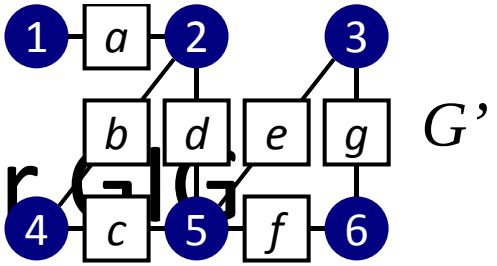



1. collect all vertices with degree =1.
2. checking neighbors, we can see ● and ○.
3. by them, we can see *V* and *E*, and we can construct *G*=(*V*,*E*).

Now, it is sufficient to show that *G''* is a GIG.

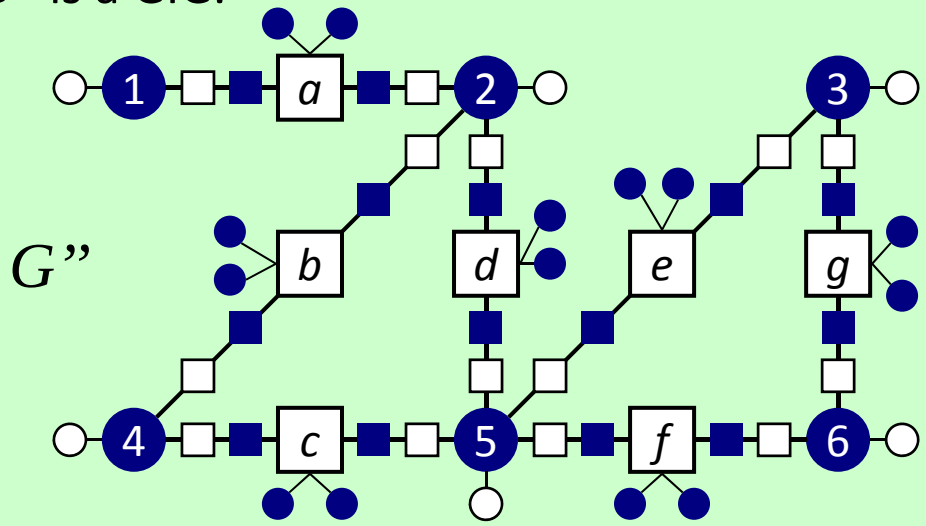
# *GI*-completeness for GIG




[Thm] Grid Intersection Graphs are *GI*-complete

*G''* is a GIG.

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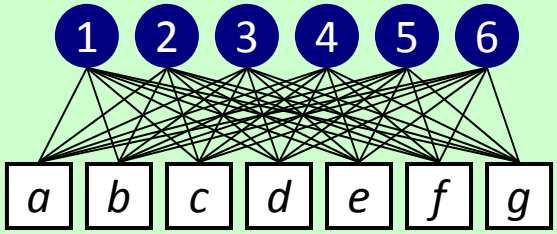
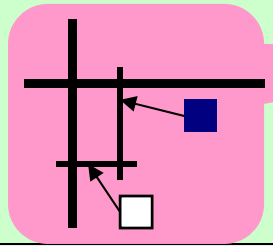
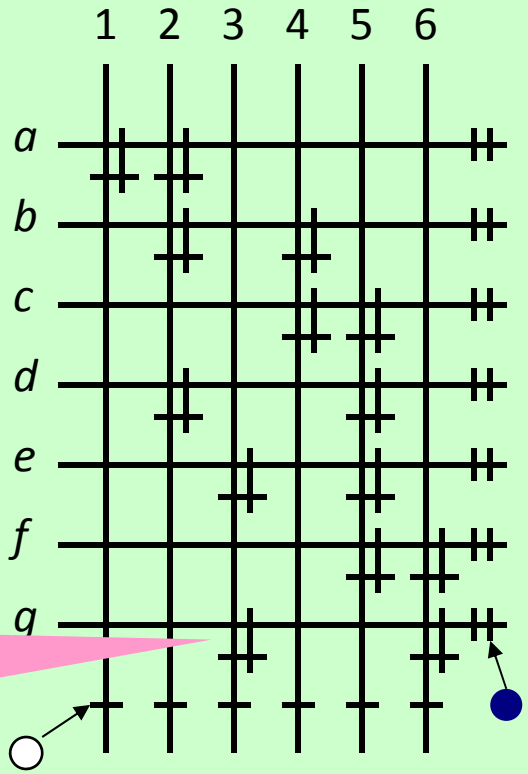




- Two gra

C.f. The recognition of UGI is NP-complete
Mustata and Pergel 2013
(arXiv:1306.1855v1)

# F e work (1).

Each vertex corresponds to an orthogonal ray

Grid inte tion graphs

Unit grid intersection graphs?

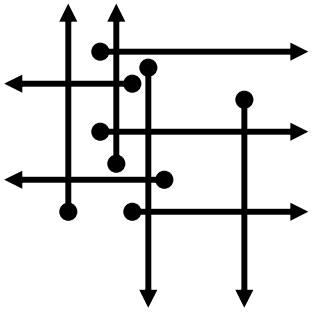
Orthogonal ray graphs?

Chordal bipartite graphs

2D-orthogonal ray graphs

Convex bipartite graphs

Bipartite permutation graphs



Poly-time solvable

GI-complete

Open

I think… they are GI-complete?: You may find nice reduction!!

# Future

Trapezoid graphs?

Triangle graphs?

Simple triangle graphs?

Permutation graphs

Interval graphs

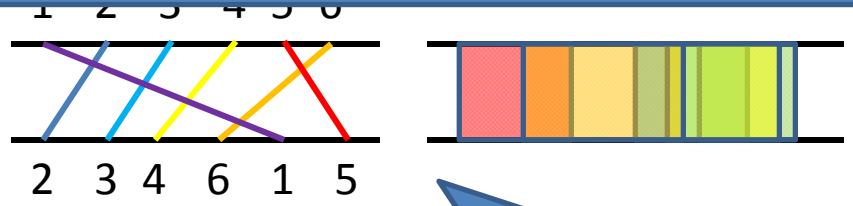
linear-time solvable

Open

Recognition is Poly-time solvable
(Mertzios and Corneil 2011)

Recognition is NP-complete!
(Mertzios 2012)

Recognition is NP-complete?
(Mertzios 2013?)




If they are poly-solvable,
You may find *canonical tree representations* of the graphs

"Multi-chain ordering" ?