Independence Polynomials of $k$-trees

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**Definition**

**Independence Polynomial of** $G$

Prodinger and Tichy, 1982  
Gutman and Harary, 1983

Let $G$ be a graph. For $s \geq 0$ let $f_s = f_s(G)$ be the number of independent vertex sets of cardinality $s$ in $G$. Then the independence polynomial of $G$, denoted ”$G(x)$”, is defined by

$$G(x) = \sum_{s \geq 0} f_s x^s$$
Properties of Coefficients of Independence Polynomials

Let $G$ be a graph with $n$ vertices and $m$ edges. Then:

i) $f_0 = 1$;

ii) $f_1 = n$;

iii) $f_2 = \binom{n}{2} - m$;

iv) For $s \geq 3$, $f_s$ is not determined by $n$ and $m$:

- $P_4(x) = 1 + 4x + 3x^2$
- $St_4^1(x) = 1 + 4x + 3x^2 + x^3$.

Examples of Independence Polynomials

- $K_n(x) = 1 + nx$
- $E_n(x) = (1 + x)^n = \sum_{s \geq 0} \binom{n}{s} x^s$
- $St_n^1(x) = x + (1 + x)^{n-1} = 1 + nx + \sum_{s \geq 2} \binom{n-1}{s} x^s$
Reduction Formulas for Independence Polynomials

i) \((G \cup H)(x) = G(x)H(x)\);

ii) \((G \oplus H)(x) = -1 + G(x) + H(x)\);

Vertex Reduction

iii) Let \(v\) be a vertex of \(G\). Let \(G_1 = G - v\) and let \(G_2 = G - N[v]\). Then \(G(x) = G_1(x) + xG_2(x)\).

Coefficient Version of Vertex Reduction

Let \(v\) be a vertex of \(G\).

\[f_s(G) = f_s(G_1) + f_{s-1}(G_2) \quad (s \geq 1)\]

Edge Reduction

iv) Let \(e\) be an edge of \(G\). Let \(G_3 = G - e\) and \(G_4 = G - (N[v] \cup N[w])\). Then \(G(x) = G_3(x) - x^2G_4(x)\).
Easy Example: $P_n$
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By vertex reduction on an end-vertex,

$$P_n(x) = P_{n-1}(x) + xP_{n-2}(x)$$
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$$\lambda_1 = \frac{1 + \sqrt{1 + 4x}}{2} \quad \lambda_2 = \frac{1 - \sqrt{1 + 4x}}{2}$$
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$$P_n(x) = c_1(x)(\frac{1+\sqrt{1+4x}}{2})^n + c_2(x)(\frac{1-\sqrt{1+4x}}{2})^n$$
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Use initial conditions: $P_0(x) = 1$ and $P_1(x) = 1 + x$.

Solving for $c_1(x)$ and $c_2(x)$ and simplifying yields:
Easy Example: $P_n$

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Use initial conditions: $P_0(x) = 1$ and $P_1(x) = 1 + x$.

Solving for $c_1(x)$ and $c_2(x)$ and simplifying yields:

$$P_n(x) = \frac{1}{\sqrt{1+4x}} \left[\left(\frac{1+\sqrt{1+4x}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{1+4x}}{2}\right)^{n+2}\right]$$

Similarly: $f_s(P_n) = \binom{n+1-s}{s}$
Theorem

Wingard, 1995

Let $T$ be a tree with $n$ vertices with $T(x) = \sum_{s \geq 0} f_s x^s$.

Then for $s \neq 1$:

$$\binom{n+1-s}{s} \leq f_s \leq \binom{n-1}{s}$$

Wingard’s bounds are sharp

$$f_s(P_n) = \binom{n+1-s}{s} \quad \text{and if } s \neq 1 \quad f_s(St_n^1) = \binom{n-1}{s}$$
A pair of non-isomorphiphic graphs with the same independence polynomial:

\[ G_1(x) = 1 + 4x + 2x^2 = G_2(x) \]
Another pair:

\[ G_3(x) = 1 + 7x + 14x^2 + 8x^3 = G_4(x) \]
$k$-trees

Definition:
(Beineke and Pippert, 1969)

(1) $K_{k+1}$ is a $k$-tree on $k + 1$ vertices;

(2) If $T$ is a $k$-tree on $n$ vertices and $C$ is a $k$-clique of $T$, then a $k$-tree on $n + 1$ vertices is formed by adjoining a new vertex $v$ to $T$ and joining $v$ by an edge to each vertex of $C$. 
Building a 2-tree
Building a 2-tree
Building a 2-tree
Building a 2-tree
Building a 2-tree
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Definition
Let $T$ be a $k$-tree and $v$ a vertex of $T$. If the neighbors of $v$ induce a clique, then $v$ is said to be a \textbf{simplicial vertex}.

Vertices 1, 6, and 8 are simplicial.

In a $k$-tree, a vertex $v$ is simplicial if and only if $\text{degree}(v) = k$. 
Some facts about $k$-trees:

Let $T$ be a $k$-tree on $n$ vertices, $n \geq k + 2$. Then:

1. $\delta(T) = k$;
2. $T$ is maximally $k$-degenerate;
3. $T$ has at least two simplicial vertices;
4. $\omega(T) = k + 1$;
5. $\chi(T) = k + 1$;
6. $T$ is uniquely $(k + 1)$-colorable;
7. $\chi_T(\lambda) = \lambda^k (\lambda - k)^{n-k}$;
8. $T$ has exactly $n - k$ $(k + 1)$-cliques and exactly $kn - \binom{k+1}{2}$ edges.
Examples of $k$-trees

**Definition**

$P_n^k$: the $k$-path on $n$ vertices with $n \geq k + 1$.

Let $v_1, v_2, \ldots v_{k+1}$ be a $(k + 1)$-clique.

For $k + 2 \leq i \leq n$, let $v_i$ be adjacent to $v_{i-1}, v_{i-2}, \ldots v_{i-k}$.
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![Diagram](attachment:image.png)
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![Diagram of a 3-path on 7 vertices](image-url)
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\begin{tikzpicture}[scale=0.8]

\node[circle, draw] (1) at (0,0) {1};
\node[circle, draw] (2) at (2,0) {2};
\node[circle, draw] (3) at (1,-2) {3};
\node[circle, draw] (4) at (3,0) {4};
\node[circle, draw] (5) at (2,-2) {5};
\node[circle, draw] (6) at (4,0) {6};
\node[circle, draw] (7) at (5,-2) {7};

\draw (1) -- (2);
\draw (1) -- (3);
\draw (1) -- (5);
\draw (2) -- (3);
\draw (2) -- (4);
\draw (3) -- (4);
\draw (3) -- (5);
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\draw (5) -- (6);
\draw (6) -- (7);
\end{tikzpicture}
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$P_{11}^3$: the 3-path on eleven vertices
$P^3_{11}$: the 3-path on eleven vertices

$P^2_{12}$: the 2-path on twelve vertices
**Definition**

$St_n^k$: the k-star on $n$ vertices with $n \geq k + 1$.

1. Let $v_1, v_2, \ldots v_k$ be a $k$-clique.
2. For $k + 1 \leq i \leq n$, let $v_i$ be adjacent to $v_1, v_2, \ldots v_k$. 
\[ St^2_8 : \text{the 2-star on eight vertices} \]
$St^2_8$ : the 2-star on eight vertices
\( St_8^2 \): the 2-star on eight vertices
$St_8^2$: the 2-star on eight vertices
$St^2_{8}$: the $2$-star on eight vertices
$St^2_{8}: \text{the 2-star on eight vertices}$
Definition

$Sp_n^k$: the $k$-spiral on $n$ vertices with $n \geq k + 2$.

1. Let $v_1, v_2, \ldots, v_{k+1}$ be a $k + 1$-clique.
2. For $k + 2 \leq i \leq n$, let $v_i$ be adjacent to $v_2, v_3, \ldots, v_k$ and $v_{i-1}$. 
$Sp^2_7$: the 2-spiral on seven vertices
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$Sp^2_7$: the 2-spiral on seven vertices
Independence Polynomials of Some $k$-trees
Song, Wei et al. 2010

\begin{align*}
\text{i)} \quad St_n^k(x) &= kx + (1 + x)^{n-k} = 1 + nx + \sum_{s \geq 2} \binom{n-k}{s} x^s \\
\text{ii)} \quad Sp_n^k(x) &= (k - 1)x + P_{n+1-k}(x) = 1 + nx + \sum_{s \geq 2} \binom{n+2-k-s}{s} x^s \\
\text{iii)} \quad P_n^k(x) &= \sum_{s \geq 0} \binom{n+k-ks}{s} x^s
\end{align*}
Generalizing Wingard’s Inequality

Theorem

Song, Wei et al. 2010
Let $G$ be a $k$-degenerate graph with $n$ vertices. Then, for $s \neq 1$:

i) \[ \binom{n+k-ks}{s} \leq f_s(G); \]

ii) If $G$ is maximal $k$-degenerate, then $f_s(G) \leq \binom{n-k}{s}$.

Corollary

Let $T$ be a $k$-tree with $n$ vertices. Then, for $s \neq 1$:

\[ \binom{n+k-ks}{s} \leq f_s(G) \leq \binom{n-k}{s} \]
A construction of well-covered graphs:

Let $G$ be a graph and $C_1, C_2, \ldots, C_r$ a collection of cliques partitioning the vertex set of $G$. For $1 \leq i \leq r$ let $H_i$ be a clique and join each vertex of $C_i$ to each vertex of $H_i$. The resulting graph is called a **Corona** over $G$. 
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A corona over $P_4$

![Graph Diagram]

$H_1$ $H_1$ $H_2$

$C_1$ $C_1$ $C_2$ $C_2$
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A corona over $P_4$
Independence Polynomials of Coronas

Theorem


Let $G$ be a graph. Let $\{C_1, C_2, \ldots, C_r\}$ be cliques partitioning the vertex set of $G$ and let $H_i = K_s$ for $1 \leq i \leq r$. If $J$ is the corona over $G$, then

$$J(x) = (1 + sx)^r G\left(\frac{x}{1+sx}\right)$$
Example

Let $G$ be $P_4$, each $C_i$ and each $H_i$ a singleton. The corona $J$ is a "comb."

\[ P_4(x) = 1 + 4x + 3x^2 \]
\[ r = 4 \]
\[ s = 1 \]

\[ J(x) = [1 + x]^4 [1 + 4\left(\frac{x}{1+x}\right) + 3\left(\frac{x}{1+x}\right)^2] \]
\[ = 1 + 8x + 21x^2 + 22x^3 + 8x^4 \]


¡MUCHAS GRACIAS, AMIGOS!