Comparing the local chromatic number of a digraph and its underlying undirected graph

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Def. (Erdős, Füredi, Hajnal, Komjáth, Rödl, Seress 1986): *The local chromatic number of graph G is*

$$\psi(G) := \min_{c} \max_{v \in V(G)} |\{c(u) : \{u, v\} \in E(G)\}| + 1,$$

where the minimization is over all proper colorings c of G.

In words: $\psi(G)$ is the minimum number of colors that must appear in the most colorful closed neighborhood of a vertex in any proper coloring.

Obviously: $\psi(G) \leq \chi(G)$. (Use only $\chi(G)$ colors.)

Thm. (EFHKRS 1986): $\forall k, \exists G : \psi(G) = 3, \chi(G) > k.$

Thm. (Körner, Pilotto, S. 2005):

$$\chi^*(G) \leq \psi(G),$$

where $\chi^*(G)$ is the fractional chromatic number of G.

Not too many graphs have $\chi^* << \chi$. Kneser graphs $\mathrm{KG}(n,k)$ and Schrijver graphs $\mathrm{SG}(n,k)$ are such graphs. A sample theorem:

Thm. (S.-Tardos 2006, S.-Tardos-Vrećica 2009): If $t := \chi(\mathrm{SG}(n,k)) = n - 2k + 2$ and n, k large enough, then

$$\psi(\mathrm{SG}(n,k)) = \lfloor t/2 \rfloor + 2.$$

Def. (Körner-Pilotto-S. 2005): The directed local chromatic number of a digraph D is

$$\psi_d(D) := \min_c \max_{v \in V(D)} |\{c(u) : (v,u) \in E(G)\}| + 1,$$

where the minimization is over all proper colorings c of D.

The novelty is that here we consider **out**-neighborhoods.

If all edges in $\vec{G}$ are present in both directions,

then $\psi_d(\vec{G}) = \psi(G)$. In general we have $\psi_d(\vec{G}) \leq \psi(G)$.

Oriented versus undirected graphs

We are interested in oriented versions of G, meaning that all edges of G are present in exactly one direction.

Def.:

$$\psi_{d,\max}(G) = \max\{\psi_d(\vec{G}) : \vec{G} \text{ is an orientation of } G\}.$$

$$\psi_{d,\min}(G) = \min\{\psi_d(\vec{G}) : \vec{G} \text{ is an orientation of } G\}.$$

Question: How do these invariants relate to $\psi(G)$?

In particular: Can $\psi_{d,\max}(G)$ be smaller than $\psi(G)$?

Thm (S.-Tardos-Zsbán): There exists a graph G with $\psi_{d,\max}(G) < \psi(G)$.

Annoyingly, the following question is open:

Can the difference between $\psi_{d,\max}(G)$ and $\psi(G)$ be arbitrarily large?

Fractional versions

Definition of fractional local chromatic number $\psi^*(G)$ and of fractional directed local chromatic number $\psi_d^*(\vec{G})$ is straightforward: consider a fractional coloring of G and look at the total weight in closed neighborhoods versus closed out-neighborhoods.

Thm. (Körner-Pilotto-S. 2005):

$$\forall G: \quad \psi^*(G) = \chi^*(G).$$

Thm. (S.-Tardos-Zsbán):

$$\max_{\vec{G}} \psi_d^*(\vec{G}) = \chi^*(G).$$

Thus the minimum of the ratio $\frac{\psi^*(G)}{\psi_d^*(\vec{G})}$ is 1 (for every G). The next result gives the maximum possible ratio.

Thm. (S.-Tardos-Zsbán):

The supremum of possible values of the ratio $\frac{\psi^*(G)}{\psi_d^*(\vec{G})}$ is e, the basis of the natural logarithm.

We give a more refined statement on the next two slides.

Thm. (S.-Tardos-Zsbán):

(a) For every finite, loopless directed graph G we have

$$\chi^*(G) \le \frac{k^k}{(k-1)^{k-1}} < ek,$$

where $k = \psi_d^*(G) > 1$ and e is the basis of the natural logarithm.

(b) For every $k \geq 2$ and $\varepsilon > 0$ there exists a finite, loopless directed graph G with $\psi_d^*(G) \leq k$ and

$$\chi^*(G) > \frac{k^k}{(k-1)^{k-1}} - \varepsilon.$$

If k is an integer, then the above graph can be chosen to further satisfy $\psi_d(G) = k$.

Remark: Shanmugam-Dimakis-Langberg independently gave a somewhat weaker (than e) upper bound. They also found this result to be relevant in the context of an information transmission problem.

In contrast to the above, we have

Thm (S.-Tardos 2011): For every $k \in N$ there exist graphs G and their orientation $\vec{G}$ with $\psi_d^*(\vec{G}) = \psi_d(\vec{G}) = 2$ and $\psi(G) > k$.

The claimed graphs are shift graphs.