Three-colourability of planar graphs without 5-cycles and triangular 3- and 6-cycles

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Joint work with Babak Farzad

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Graph Colouring; An Introduction

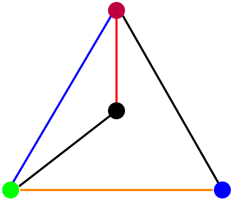

Figure : A colouring of vertices of a graph.

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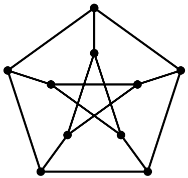

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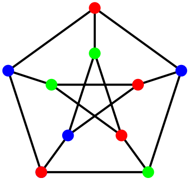

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History:

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2. Three-colour theorem: [Grötzsch; 1959] If G is planar and triangular free, then $\chi(G) \leq 3$.

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2. Relaxation of Steinberg conjecture: [Erdős; 1990] Find the smallest C such that a $\{4, \ldots, C\}$-cycle-free planar graph is 3-colourable.

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4. [Borodin et al.; 2005] $\Rightarrow$ $\{4, \ldots, 7\}$-cycle-free planar graphs.

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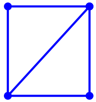
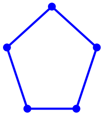
2. [Borodin et al.; 2010] Planar graphs without triangles adjacent to cycles of length from 4 to 7 are 3-colorable.

Claim: Graphs without the following configurations are 3-colourable:



F_1



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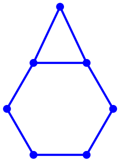

F_3

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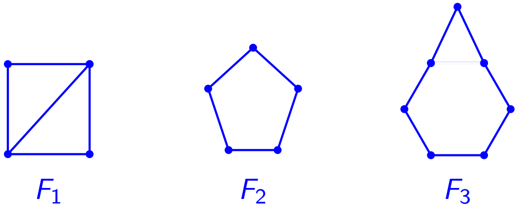

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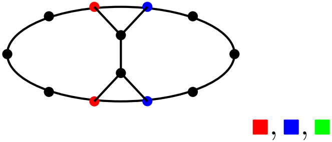

Figure : A colouring of 10-cycle that cannot be extended to *d*-claw.

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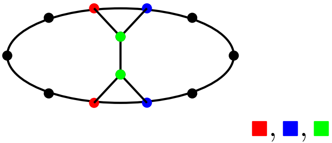

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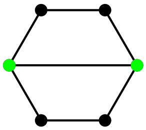
Bad cycles:

1. 6-cycle: it's internal face is partitioned into 4-cycles.



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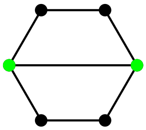

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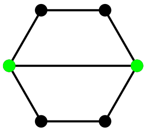

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2. 9-cycle ⇒ one 7-cycle and one or more 4-cycles.
3. 10-cycle ⇒ Either a *d*-claw or one 8-cycle and one or more 4-cycles.

Main theorem: Any 3-colouring of the boundary of the exterior face D, which is a good cycle, of any planar graph without F_1, F_2, and F_3 can be extended to a 3-colouring of the graph.

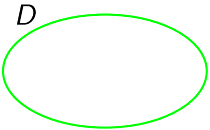

Figure : The outer boundary of the external face of G.

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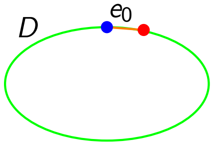

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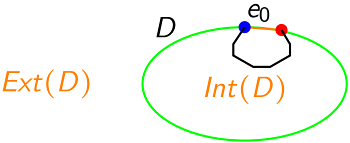

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(4) G has no separating good cycle; ($Int(C) \neq \emptyset$ and $Out(C) \neq \emptyset$).

S_i: separating cycle of length i.

(5) If a good cycle C in G has an internal chord e, then

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(7) If C is good, then there is no 2-path xyz joining two

non-consecutive vertices of C through $y \in Int(C)$.

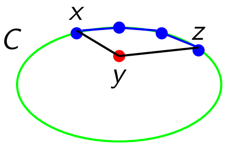

Figure : No 2-path joining non-consecutive vertices of a good cycle C.

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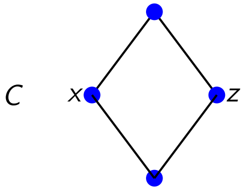

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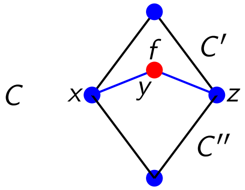

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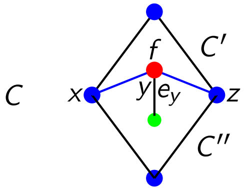

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Proof: ... Continued...

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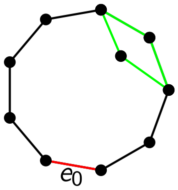
   (a) $|C| = 9$ and C is stretched $\Rightarrow$ e_0 is on C''

   $\Rightarrow$ C'' cannot have a chord (forming F_i or C is bad)

   $\Rightarrow$ C'' is an S_9 (bad) with bad partition P

   $\Rightarrow$ $P \cup \{f\}$: a bad partition of C (Contradiction)



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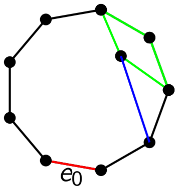

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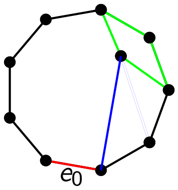

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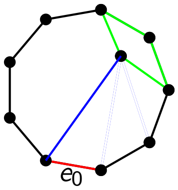

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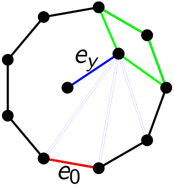

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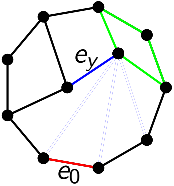

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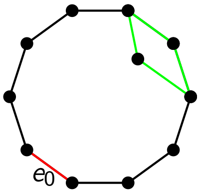
Proof: ... Continued...

3. $|C''| = 10$: $\Rightarrow$ e_0 is on C'' $\Rightarrow$ C'' cannot have a chord

$\Rightarrow$ C'' is an S_{10} (bad with partition P or d-claw)

$\Rightarrow$ If d-claw, then e_y is on a triangle adjacent to f; F_2

(Contradiction)



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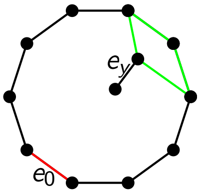

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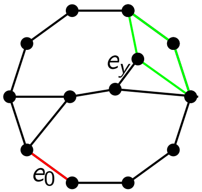

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Excluding certain configurations: By transforming G into a smaller graph G', and in doing so we make sure not to:

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Next:

(i) The colouring of D cannot be extended to G' (contradiction),

(ii) The colouring of D can be extended to G (contradiction).

Properties of G ... Continued...

(8) G has no 4-cycle other than D.

Sketch of proof: (By contradiction) If $wxyz \neq D$ is a 4-cycle in G:

(i) G has no separating 4-cycle and $F_1 \quad \Rightarrow \quad wxyz$ is a face,

(ii) D has no chord $\quad \Rightarrow \quad$ not all w, x, y, z are on D; let $y \in Int(D)$,

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Properties of G ... Continued...

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Proof: Similar to (8).

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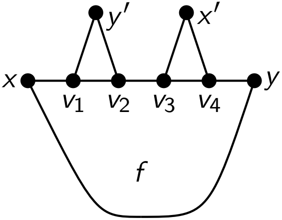

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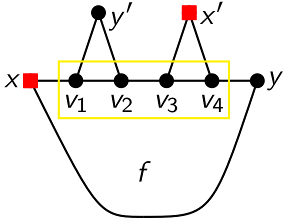

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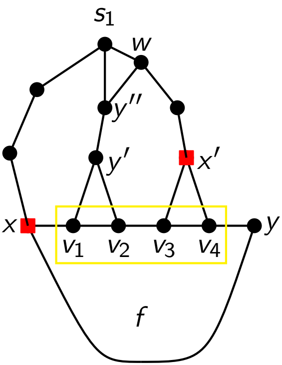

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(10) G has no internal tetrad.

Proof:

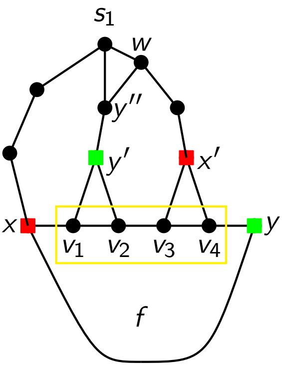

Figure : No tetrad.

(10) G has no internal tetrad.

Proof:

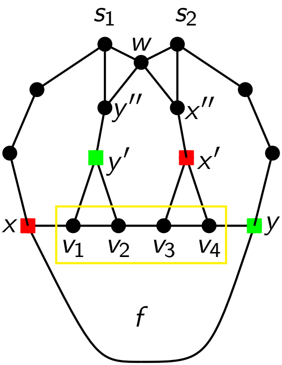

Figure : No tetrad.

No internal tetrad.... Continued

d(w)=4: the colouring can be extended.

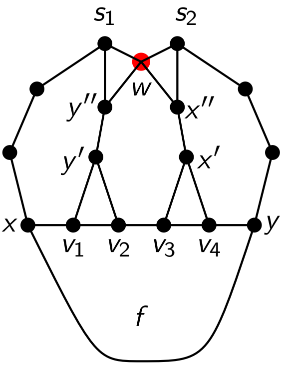

Figure : No tetrad.

No internal tetrad.... Continued

d(w)=4: the colouring can be extended.

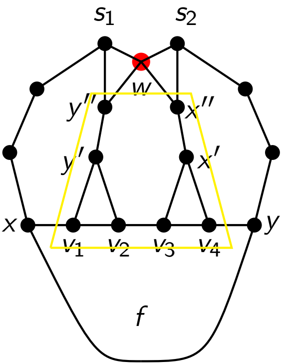

Figure : No tetrad.

No internal tetrad.... Continued

$d(w) \geq 5$:

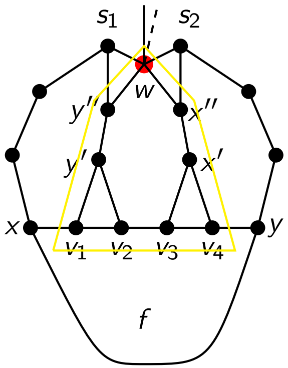

Figure : No tetrad.

No internal tetrad.... Continued

$d(w) \geq 5$:

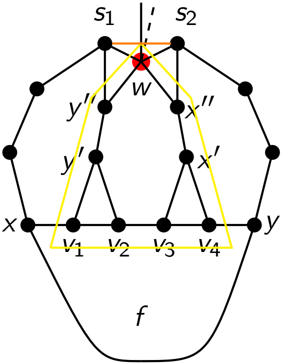

Figure : No tetrad.

(11) G has at most one M-face and no MM-faces.

Proof:

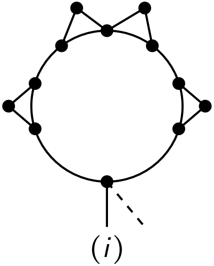

Figure : (i) M-face and (ii) MM-face.

Obstacle: Making D a d-claw.

(11) G has at most one M-face and no MM-faces.

Proof:

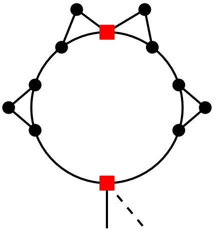

Figure : (i) M-face and (ii) MM-face.

Obstacle: Making D a d-claw.

(11) G has at most one M-face and no MM-faces.

Proof:

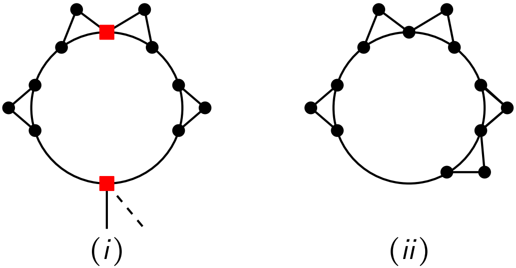

Figure : (i) M-face and (ii) MM-face.

Obstacle: Making D a d-claw.

(11) G has at most one M-face and no MM-faces.

Proof:

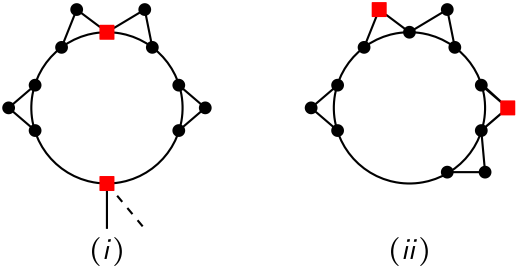

Figure : (i) M-face and (ii) MM-face.

Obstacle: Making D a d-claw.

(11) G has at most one M-face and no MM-faces.

Proof:

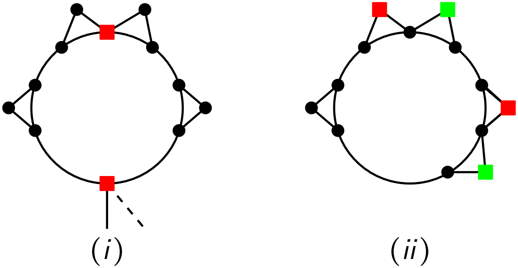

Figure : (i) M-face and (ii) MM-face.

Obstacle: Making D a d-claw.

(12) G does not have the following configurations.

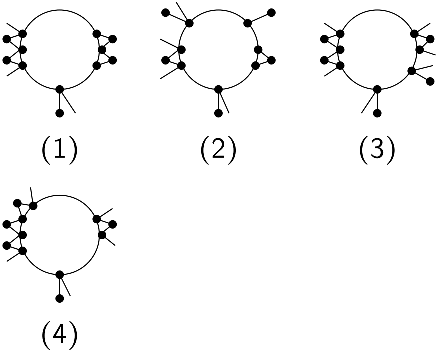

Figure : Bad 7-faces.

Proof (case (4)):

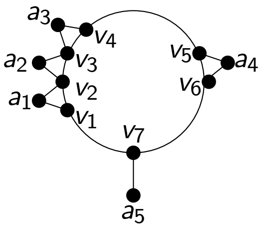

Figure : Bad 7-face (4).

Proof (case (4)):

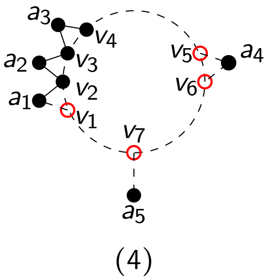

Figure : Bad 7-face (4).

Proof (case (4)):

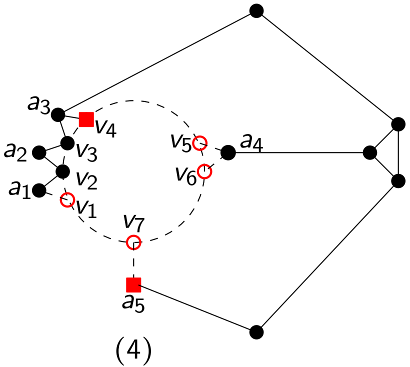

Figure : Bad 7-face (4).

Proof (case (4)):

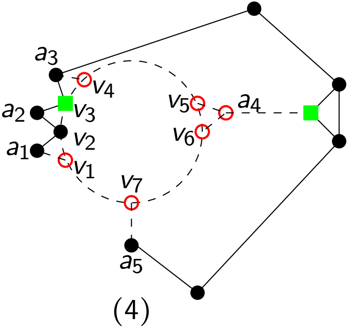

Figure : Bad 7-face (4).

Proof (case (4)):

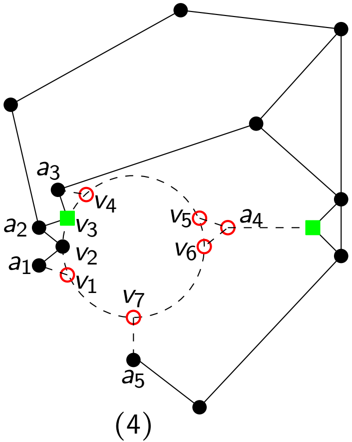

Figure : Bad 7-face (4).

Theorem: The properties of G are incompatible.

Proof: Using discharging method.

Corollary: The planar graphs without F_1, F_2, and F_3 are 3-colourable.

Thank You!