Three-colourability of planar graphs without 5-cycles and triangular 3- and 6-cycles

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Figure: A colouring of vertices of a graph.
Proper graph colouring: Assignments of colours to the vertices of a graph such that no two adjacent vertices are coloured the same.
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Chromatic number: The smallest number of colours needed to properly colour the vertices of a graph $G$; $\chi(G)$.

Example:

Figure: $\chi(P) = 3$. 
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Chromatic number: The smallest number of colours needed to properly colour the vertices of a graph $G$; $\chi(G)$.

Example:

![Graph diagram with vertices coloured]

Figure: $\chi(P) = 3$. 
History:

1. **Four-colour theorem**: [Appel-Haken; 1977] If $G$ is planar, then $\chi(G) \leq 4$; every plane map is 4-colorable.
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1. **Four-colour theorem**: [Appel-Haken; 1977] If $G$ is planar, then $\chi(G) \leq 4$; every plane map is 4-colorable.

2. **Three-colour theorem**: [Grötzsch; 1959] If $G$ is planar and triangular free, then $\chi(G) \leq 3$. 
Three-colourability of planar graphs:

1. Steinberg conjecture: [1976] Every $\{4, 5\}$-cycle-free planar graph is 3-colourable.
Three-colourability of planar graphs:

1. **Steinberg conjecture:** [1976] Every \(\{4, 5\}\)-cycle-free planar graph is 3-colourable.

2. **Relaxation of Steinberg conjecture:** [Erdős; 1990] Find the smallest \(C\) such that a \(\{4, \ldots, C\}\)-cycle-free planar graph is 3-colourable.
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2. [Borodin; 1996] \(\Rightarrow\) \(\{4, \ldots, 10\}\)-cycle-free planar graphs.

3. [Borodin; 1996 (also, Sanders-Zhou; 1995)]

\(\Rightarrow\) \(\{4, \ldots, 9\}\)-cycle-free planar graphs.
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   $\Rightarrow \{4, \ldots, 9\}$-cycle-free planar graphs.

4. [Borodin et al.; 2005] $\Rightarrow \{4, \ldots, 7\}$-cycle-free planar graphs.
More results:

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1. [Borodin et al.; 2009] Planar graphs without \{5, 7\}-cycles and adjacent triangles are 3-colorable.

2. [Borodin et al.; 2010] Planar graphs without triangles adjacent to cycles of length from 4 to 7 are 3-colorable.
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\[ F_1 \quad F_2 \quad F_3 \]

.... Proof follows ...
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**A \( d \)-claw:**

![Diagram of a \( d \)-claw]

**Figure:** A colouring of 10-cycle that cannot be extended to \( d \)-claw.
**Stretched edge:** An edge that is not on a \(\{4, 6\}\)-cycle.

**A \(d\)-claw:**

![Diagram of a 10-cycle with a highlighted edge]

**Figure:** A colouring of 10-cycle that cannot be extended to \(d\)-claw.
Bad cycles:

1. 6-cycle: it’s internal face is partitioned into 4-cycles.

![Diagram of a 6-cycle](image)

Figure: Bad 6-cycle.
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2. 9-cycle $\Rightarrow$ one 7-cycle and one or more 4-cycles.
Bad cycles:

1. 6-cycle: it’s internal face is partitioned into 4-cycles.

2. 9-cycle $\Rightarrow$ one 7-cycle and one or more 4-cycles.

3. 10-cycle $\Rightarrow$ Either a $d$-claw or one 8-cycle and one or more 4-cycles.

Figure: Bad 6-cycle.
**Main theorem:** Any 3-colouring of the boundary of the exterior face $D$, which is a good cycle, of any planar graph without $F_1, F_2,$ and $F_3$ can be extended to a 3-colouring of the graph.

![Diagram](image.png)

**Figure:** The outer boundary of the external face of $G$.

**Good cycle:** Not bad and either $|C| \in \{3, 4, 6, 7\}$ or $|C| \in \{8, 9, 10\}$ and $C$ is stretched.
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Proof: (By contradiction)

1. $G$: counterexample with the fewest vertices,
2. $\phi$: a colouring of $D$ that cannot be extended to $G$. 
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2. $\phi$: a colouring of $D$ that cannot be extended to $G$.

Properties of the minimum counterexample:

(1) If $v \in \text{Int}(D)$, then $D$ does not become bad in $G – v$. 
Proof: (By contradiction)

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Properties of the minimum counterexample:

(1) If $\nu \in Int(D)$, then $D$ does not become bad in $G - \nu$.

(2) If $\nu \in Int(D)$, then $d(\nu) \geq 3$. 
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(1) If $v \in \text{Int}(D)$, then $D$ does not become bad in $G - v$.

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(3) $G$ is 2-connected.
Proof: (By contradiction)

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(2) If \( v \in \text{Int}(D) \), then \( d(v) \geq 3 \).

(3) \( G \) is 2-connected.

(4) \( G \) has no separating good cycle; \( \text{Int}(C) \neq \emptyset \) and \( \text{Out}(C) \neq \emptyset \).

\( S_i \): separating cycle of length \( i \).
(5) If a good cycle $C$ in $G$ has an internal chord $e$, then $|C| \in \{8, 9, 10\}$ and $e$ is triangular.

(6) $D$ has no chords.
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(6) $D$ has no chords.

(7) If $C$ is good, then there is no 2-path $xyz$ joining two non-consecutive vertices of $C$ through $y \in \text{Int}(C)$.

Figure: No 2-path joining non-consecutive vertices of a good cycle $C$. 
**Sketch of proof:** (By contradiction) Assume that $C$ is split by such a 2-path into cycles $C'$ and $C''$; $4 \leq |C'| \leq |C''| \leq 10$.

(i) $|C'| \leq 7$, 

(ii) If $C$ is stretched then $|C| \geq 8$ and $e_0$ lies on $C''$. 

Case $|C'| = 4$: 

1. $|C''| = 4$: Will have an $S_4$ (Contradiction).
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Figure: $|C'| = |C''| = 4$. 
Proof: ... Continued...

2. $|C''| = 9$:

(a) $|C| = 9$ and $C$ is stretched $\Rightarrow e_0$ is on $C''$

$\Rightarrow C''$ cannot have a chord (forming $F_i$ or $C$ is bad)

$\Rightarrow C''$ is an $S_9$ (bad) with bad partition $P$

$\Rightarrow P \cup \{f\}$: a bad partition of $C$ (Contradiction)

Figure: $|C'| = 4, |C''| = 9.$
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Proof: ... Continued...

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Figure: $|C'| = 4, |C''| = 9$. 
Proof: ... Continued...

3. $|C''| = 10$: $\Rightarrow$ $e_0$ is on $C''$ $\Rightarrow$ $C''$ cannot have a chord

$\Rightarrow$ $C''$ is an $S_{10}$ (bad with partition $P$ or $d$-claw)

$\Rightarrow$ If $d$-claw, then $e_y$ is on a triangle adjacent to $f$; $F_2$

(Contradiction)

Figure: $|C'| = 4, |C''| = 10.$
Proof: ... Continued...

3. \(|C''| = 10: \implies e_0 \text{ is on } C'' \implies C'' \text{ cannot have a chord}

\implies C'' \text{ is an } S_{10} \text{ (bad with partition } P \text{ or } d\text{-claw)}

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Figure: \(|C'| = 4, |C''| = 10\).
Excluding certain configurations: By transforming $G$ into a smaller graph $G'$, and in doing so we make sure not to:

(a) create loops, multiple edges or $F_1$, $F_2$, or $F_3$, 

Next:

(i) The colouring of $D$ cannot be extended to $G'$ (contradiction),

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Excluding certain configurations: By transforming $G$ into a smaller graph $G'$, and in doing so we make sure not to:

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(d) make $D$ a bad cycle (including creating $\leq 6$-cycle on $e_0$).
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Next:

(i) The colouring of $D$ cannot be extended to $G'$ (contradiction),

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Properties of $G$ ... Continued...

(8) $G$ has no 4-cycle other than $D$.

Sketch of proof: (By contradiction) If $wxyz \neq D$ is a 4-cycle in $G$:

(i) $G$ has no separating 4-cycle and $F_1 \Rightarrow wxyz$ is a face,

(ii) $D$ has no chord $\Rightarrow$ not all $w, x, y, z$ are on $D$; let $y \in Int(D),$

(iii) identify $w$ and $y$. 

(9) $G$ has no bad cycle unless possibly $d$-claws.

(10) $G$ has no 6-cycle other than $D$.

Proof: Similar to (8).
Properties of $G$ ... Continued...

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(10) $G$ has no 6-cycle other than $D$.

Proof: Similar to (8).
(10) $G$ has no internal tetrad.

Proof:

\[ \begin{array}{c}
  x \\
  v_1 \\
  v_2 \\
  v_3 \\
  v_4 \\
  y
\end{array} \]

Figure: No tetrad.
(10) $G$ has no internal tetrad.

Proof:

\begin{figure}
\centering
\includegraphics{tetrad.png}
\caption{No tetrad.}
\end{figure}
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Figure: No tetrad.
No internal tetrad.... Continued

d(w)=4: the colouring can be extended.

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No internal tetrad.... Continued

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Figure : No tetrad.
No internal tetrad.... Continued

\[ d(w) \geq 5: \]

\[ f \]

Figure: No tetrad.
No internal tetrad.... Continued

\[ d(w) \geq 5: \]

![Diagram](image)

**Figure**: No tetrad.
(11) $G$ has at most one M-face and no MM-faces.

Proof:

Figure: (i) M-face and (ii) MM-face.

Obstacle: Making $D$ a $d$-claw.
(11) $G$ has at most one M-face and no MM-faces.

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![Diagram showing M-face and MM-face](image)

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Figure: (i) M-face and (ii) MM-face.

Obstacle: Making $D$ a $d$-claw.
(12) $G$ does not have the following configurations.

Figure: Bad 7-faces.
Proof (case (4)):

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\[
\begin{align*}
&v_5 a_4 v_6 v_7 a_5 v_1 a_2 v_3 a_3 v_4 (4) \\
\end{align*}
\]

Figure: Bad 7-face (4).
Theorem: The properties of $G$ are incompatible.

Proof: Using discharging method.

Corollary: The planar graphs without $F_1$, $F_2$, and $F_3$ are 3-colourable.
Thank You!