Cycle-continuous mappings – order structure

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Outline

Nice open problem

Problem (Cycle Double Cover [Seymour, Szekerés, Tutte?])

For every bridgeless graph G exists a list cycles $C_1, \ldots, C_k$ such that every edge of G is in exactly two of them.

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An approach

Definition (Jaeger 1980/DeVos, Nešetřil, Raspaud 2006)

$G, H \ldots$ graphs
$f : E(G) \to E(H) \ldots$ mapping
f is *cycle-continuous* (cc) iff for every cycle C in H the preimage $f^{(-1)}(C)$ is a cycle in G. The existence of some cycle-continuous mapping from G to H is denoted by $G \xrightarrow{cc} H$.

- For a cubic graph $G \xrightarrow{cc} K_2^3$ iff G admits a 3-edge-coloring
- Let a mapping $f : E(G) \to E(H)$ be such that for each vertex v of G, it maps all edges incident with v to all edges incident with some vertex of H. Then f is cycle-continuous.

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Conjecture (Jaeger 1981)

Every bridgeless graph has a cycle continuous mapping to the Petersen graph.

- true for all 3-edge-colorable graphs
- true for all graphs up to 36 vertices
- and many others (e.g., for all Flower snarks)

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Is there an infinite family of bridgeless graphs such that there is no cycle-continuous mapping between any two of them?

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Every countable poset can be represented by a family of bridgeless graphs and existence of cycle-continuous mapping between them.

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Plan of the proof

- in general the cycle-continuous mapping behaves very erratically
- we tame it by using specially crafted graphs to make it behave like homomorphisms
- critical snarks are crucial for this

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Snarks

From now on we deal with cubic graphs only.

- bridgeless graph G is a *snark* if it is not 3-edge-colorable; equivalently, if $G \overset{cc}{\not\to} K_2^3$
- a snark G is *critical* if for every edge e we have $G - e \overset{cc}{\to} K_2^3$; equivalently, $\overline{G - e}$ is not 3-edge-colorable ($\overline{H}$ denoting H with suppressed vertices of degree 2) [DeVos, Nešetřil, Raspaud; da Silva, Lucchesi; Nedela, Škoviera]
- example: Petersen graph, Blanuša snarks on 18 vertices

Theorem (DNR 2006)

Suppose G, H are critical snarks, cyclically 4-edge-connected, $|E(G)| = |E(H)|$.
Then $G \overset{cc}{\to} H$ iff $G \cong H$.
Moreover, all cycle-continuous mappings from G to H are induced by the isomorphism.

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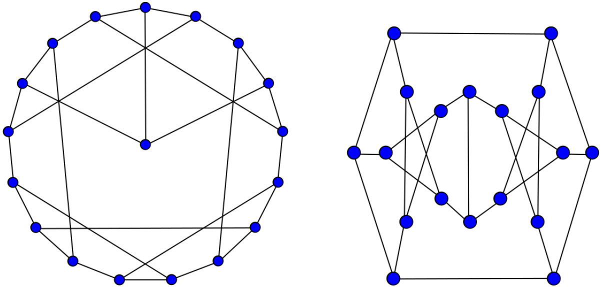
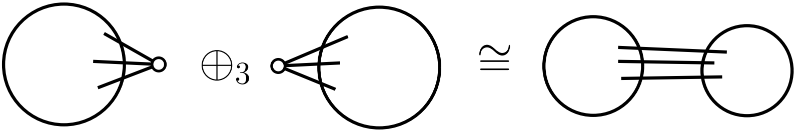
Critical snarks

Lemma

There are two snarks B_1, B_2 with 18 vertices, that are critical and nonisomorphic. Moreover, none of B_1, B_2 is vertex transitive; in particular, there is no isomorphism $f : V(B_2) \rightarrow V(B_2)$ for which $f(a) = b$.



Snark constructions: 3-join



Lemma

For any graphs G_1, G_2 we have $G_i \xrightarrow{cc} G_1 \oplus_3 G_2$ for $i = 1, 2$.

Lemma

Let G_1, G_2 be graphs.
Let K be a cyclically 4-edge-connected cubic graph that is 2-transitive.
Then $G_1 \oplus_3 G_2 \xrightarrow{cc} K$ if and only if $G_1 \xrightarrow{cc} K$ and $G_2 \xrightarrow{cc} K$.

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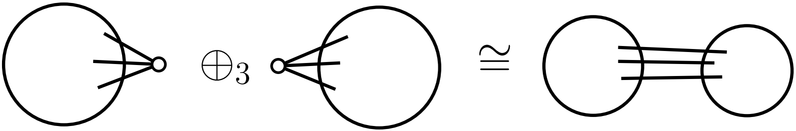

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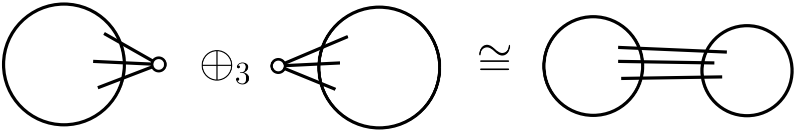

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Corollary

Let G_1, G_2 be cubic bridgeless graphs. Then $G_1 \oplus_3 G_2$ is a snark, iff at least one of G_1, G_2 is a snark.

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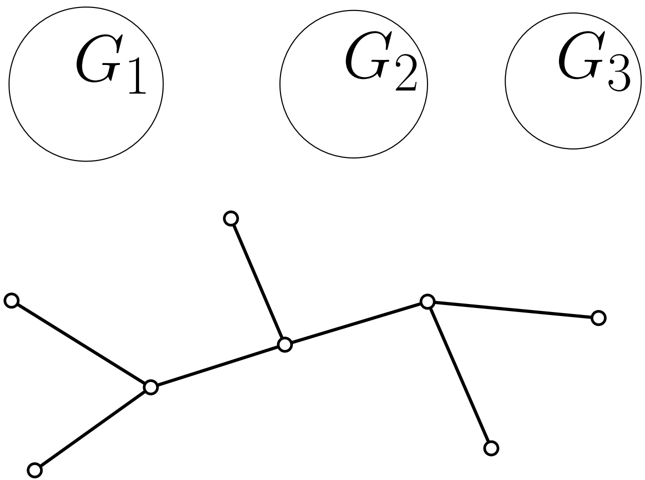
Outline

1. Introduction
2. Our results
   - Snarks
   - Tree of snarks
3. Future work

Tree of snarks – construction



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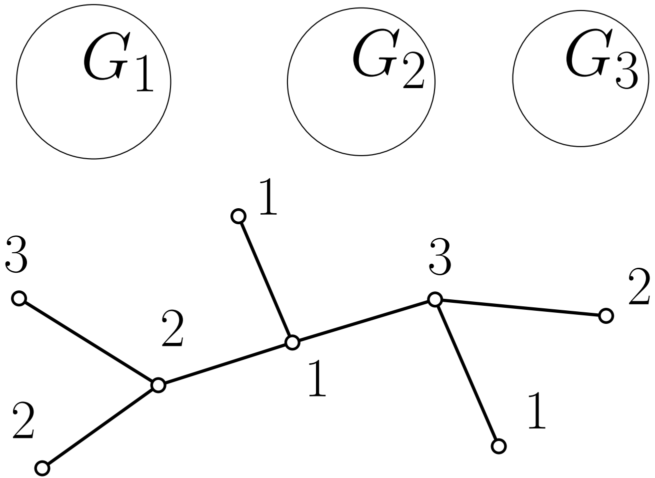

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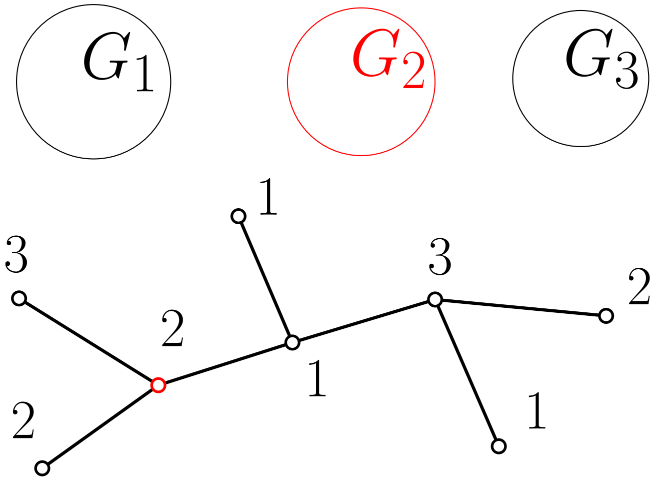

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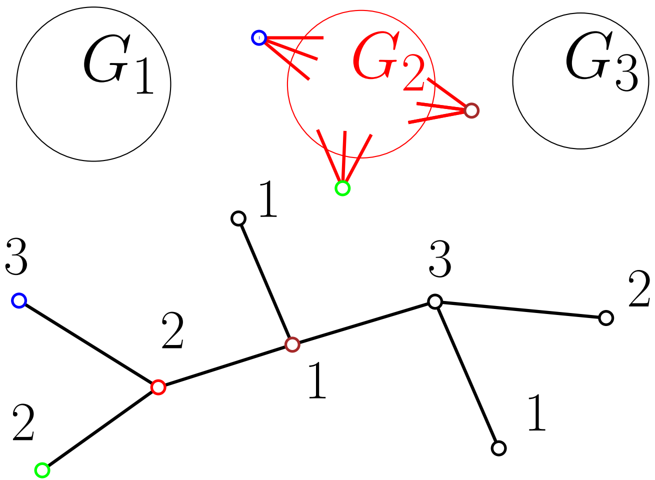

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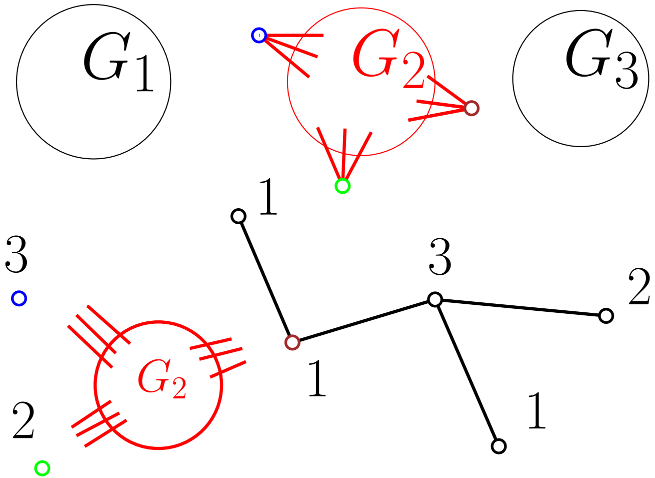

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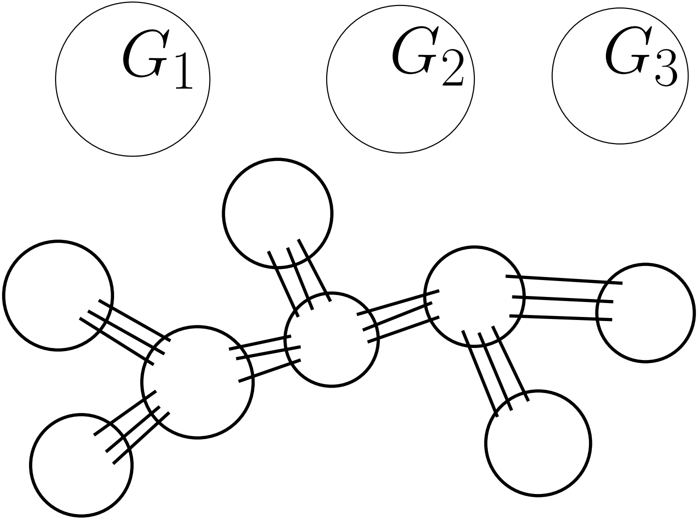

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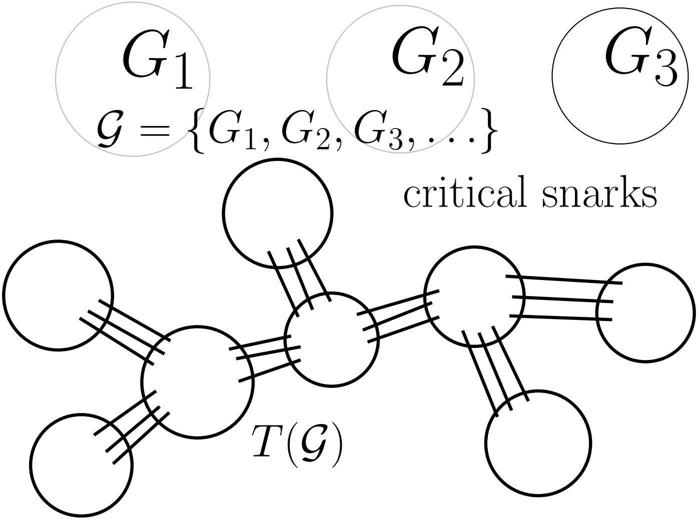

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$\mathcal{G}$ – critical snarks, cyclically 4-edge-connected,
pairwise nonisomorphic, all of the same size
$H \in T(\mathcal{G})$ and $K \in \mathcal{G}$
Then

$$K \xrightarrow{cc} H \Leftrightarrow K \cong G_i \text{ for some } G_i \in \mathcal{G} \text{ s.t. } i \text{ used on } T$$

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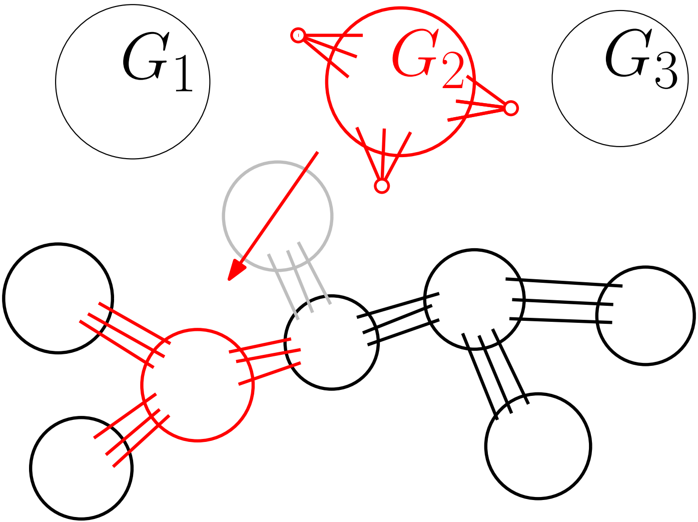

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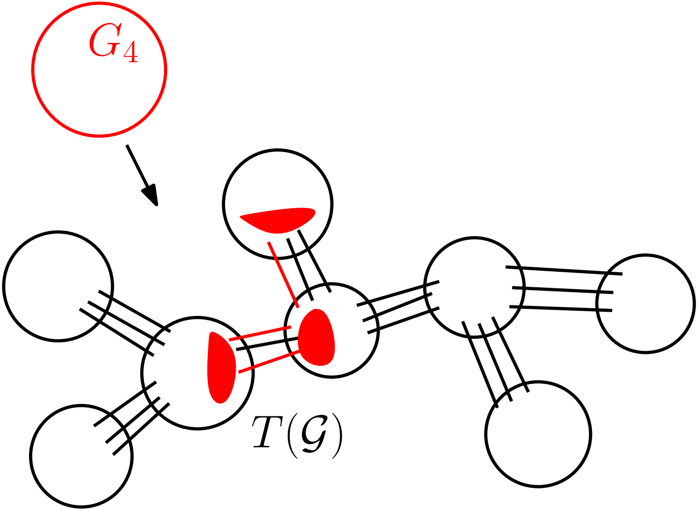

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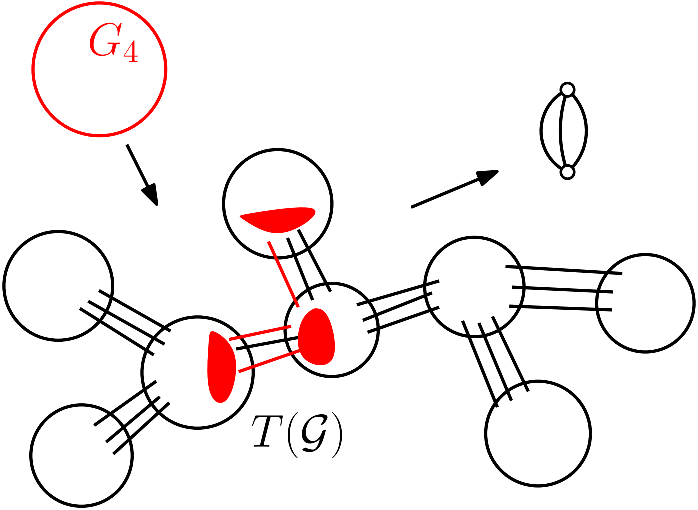

Tree of snarks – properties

Theorem

$T_1, T_2 \ldots$ trees
$c_i : V(T_i) \to [n]$, $H_i \in T_i(\mathcal{G})$ $(i = 1, 2)$
Every $g : H_1 \xrightarrow{cc} H_2$ is guided by a homomorphism $f : T_1 \to T_2$ of reflexive colored graphs: $\exists f : V(T_1) \to V(T_2)$ such that

- *$c_2(f(v)) = c_1(v)$ f respects colors, and*
- *if uv is an edge of T_1, then $f(u)f(v)$ is an edge of T_2 or $f(u) = f(v)$.*

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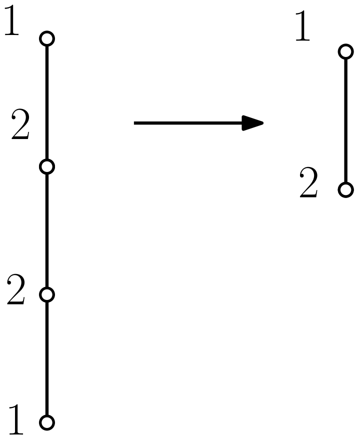

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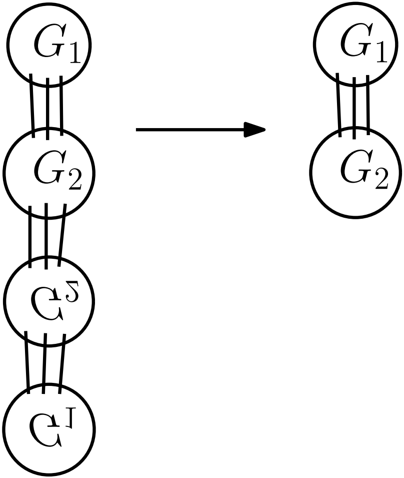

Infinite antichain

Theorem (Š. 2012)

There is an infinite family of bridgeless graphs such that there is no cycle-continuous mapping between any two of them.

Proof:

- $\mathcal{G} = \{B_1, B_2\}$, fix $a, b \in V(B_2)$ so that no automorphism maps $a \mapsto b$
- $T_n =$ a path colored as $1(2)^{n-1}1$
  $G_n \in T_n(\mathcal{G})$, taking always "a on the left, b on the right"
- Due to the choice of a, b, the "folding" in the previous figure is not possible.
- Thus, $\{G_n, n \in \mathbb{N}\}$ is an antichain.

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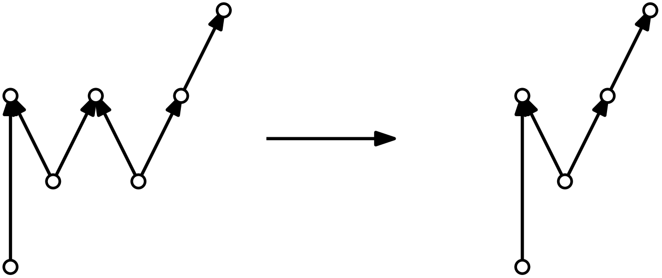
Another ingredience

Theorem (Hubička, Nešetřil 2005)

Arbitrary countable poset can be represented by finite directed paths and existence of homomorphisms between them.



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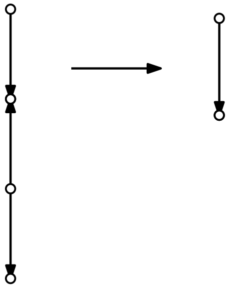

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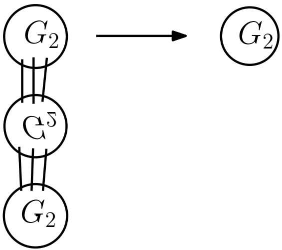

Open problems

- Is there an infinite antichain in cc mappings using cubic, cyclically 4-edge-connected graphs?
- Are there gaps in the poset of cc mappings? I.e., are there G, H s.t. $G \xrightarrow{cc} H$ but for no K we have $G \xrightarrow{cc} K \xrightarrow{cc} H$ unless $K \xrightarrow{cc} G$ or $H \xrightarrow{cc} K$?
  $K_2^3 \xrightarrow{cc} \mathrm{Pt}$ is not a gap – for example $K_2^3 \prec_{cc} B_1 \prec_{cc} \mathrm{Pt}$
- Not to forget the original question: G cubic bridgeless $\Rightarrow G \xrightarrow{cc} \mathrm{Pt}$?
  If G is a minimal counterexample, can G contain a 4-cycle?

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