A Factorization Theorem for m-rook Placements

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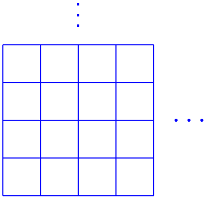
Outline

Consider the board gotten by tiling the first quadrant with squares

$\mathcal{Q} =$ 

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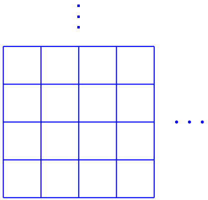
$\mathcal{Q} =$ 

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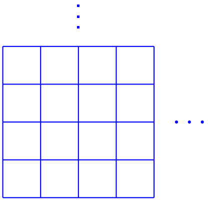
$\mathcal{Q} =$ 

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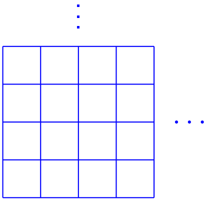
$\mathcal{Q} =$ 

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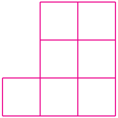
$B = (1, 3, 3) =$ 

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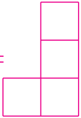
$B = (1, 3) =$  $\quad r_0(B) = 1,\ r_1(B) = 4,\ r_2(B) = 2.$

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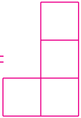
$B = (1,3) =$  $\quad r_0(B) = 1,\ r_1(B) = 4,\ r_2(B) = 2.$

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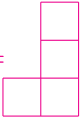
$B = (1, 3) =$ 

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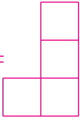
$B = (1, 3) =$  $\quad r_0(B) = 1,\ r_1(B) = 4,\ r_2(B) = 2.$

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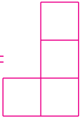
$B = (1,3) =$  $\quad r_0(B) = 1,\ r_1(B) = 4,\ r_2(B) = 2.$

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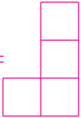
$B = (1, 3) =$  $\quad r_0(B) = 1,\ r_1(B) = 4,\ r_2(B) = 2.$

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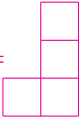
$B = (1, 3) =$  $\quad r_0(B) = 1,\ r_1(B) = 4,\ r_2(B) = 2.$

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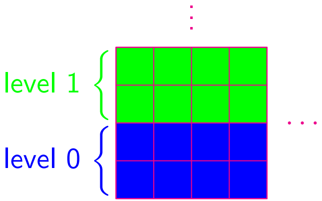

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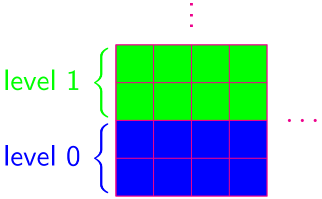

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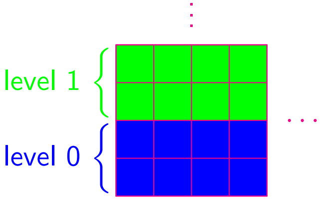

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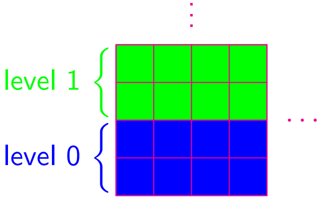

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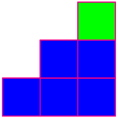
$B = (1, 2, 3) =$ 

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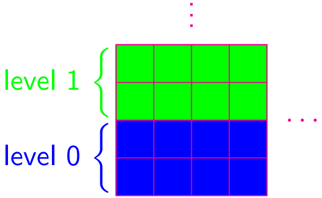

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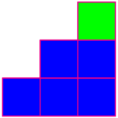

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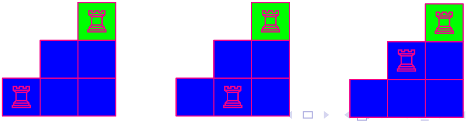

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Theorem (Briggs-Remmel)

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Then

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Theorem (Barrese-Loehr-Remmel-S)

Let $B = (b_1, \dots, b_n)$ be any Ferrers board. Then

$$\sum_{k=0}^{n} r_{k,m}(B) x\!\downarrow_{n-k,m} = \prod_{j=1}^{n} (x + \lfloor b_j \rfloor_m - (j-1)m + \epsilon_j)$$

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Ex. Suppose $m = 3$ and $B = (1, 1, 2, 3, 5, 7)$

$\therefore z_0 = (1, 1, 2),\ z_1 = (3, 5),\ z_2 = (7).$

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Note: Our theorem implies both Goldman-Joichi-White and Briggs-Remmel.

Outline

The Factorization Theorem

m-rook Placements

Arbitrary Ferrers Boards

Other Work

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for certain constants $f_{k,m}(B)$ and ask what they count. The $f_{k,m}(B)$ are weight generating functions for certain file placements (where two rooks can occupy the same row but not the same column) as shown by Haglund.

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So Ferrers boards B and B' are equivalent iff

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One can also ask for the number of elements in an equivalence class and it turns out to be a product of binomial coefficents. We have analogous results in our setting.

THANKS FOR LISTENING!