Polyhedral gluings of outerplanar graphs

David Richter

Western Michigan University

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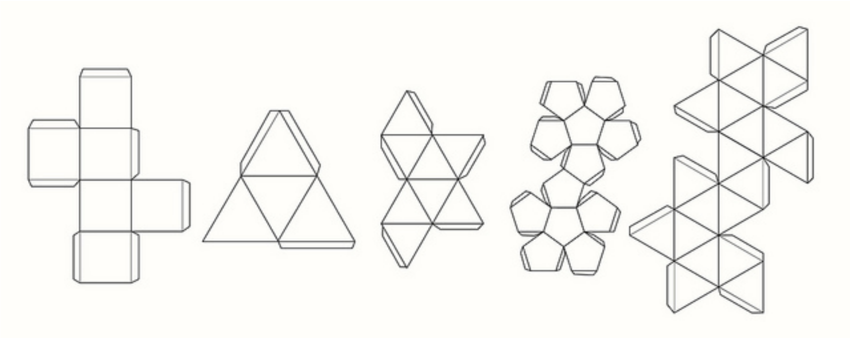
Nets of Polyhedra



The Problem of Dürer, Shephard, et al

Conjecture

Every convex polyhedron has a non-overlapping edge unfolding.

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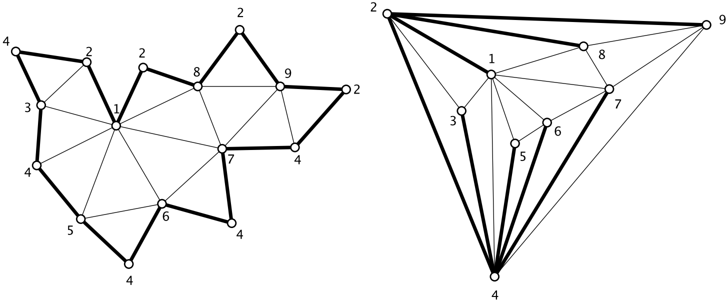
Notation: $\Delta(G)$ is the maximum degree of G.

Example



The Main Problem

Characterize outerplanar graphs which have at least one polyhedral gluing.

Polyhedral Gluings

Theorem (Steinitz)

A graph is the edge skeleton of a convex polyhedon iff it is simple, planar, and 3-connected.

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Definition

A graph is “polyhedral” if it is simple, planar, and 3-connected.

Spherical Gluings

Proposition

If G is an outerplanar graph with $|V(G)| = 2n$ and no cut vertices, then G has precisely

$$C_n = \frac{(2n)!}{n!(n+1)!}$$

spherical gluings.

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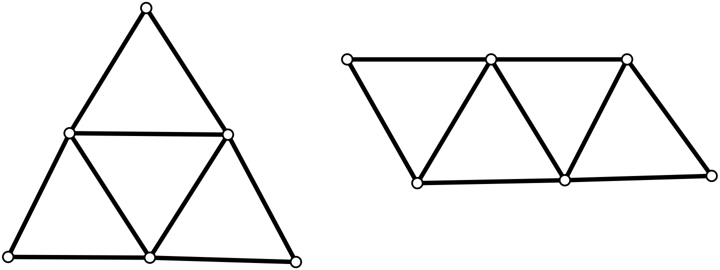
spherical gluings.

Draw a picture.

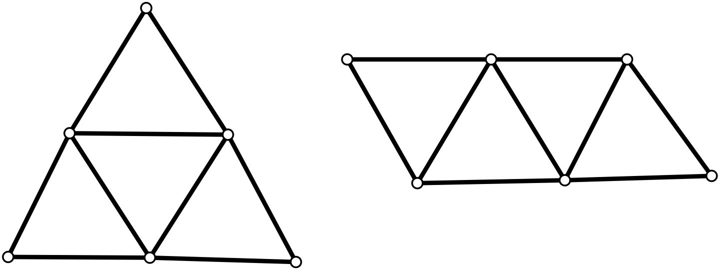
The Main Problem (restated)

Find the outerplanar graphs for which the set of spherical gluings contains at least one polyhedral gluing.

Does it glue?

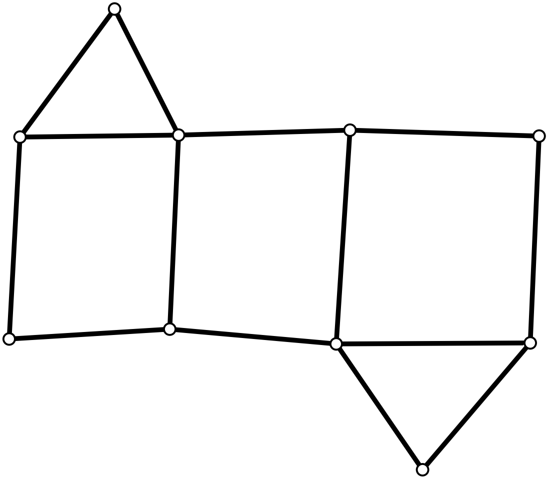


Does it glue?

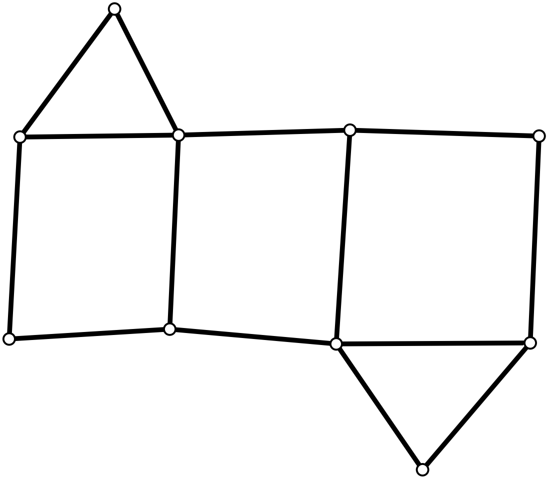


Both do.

Does it glue?

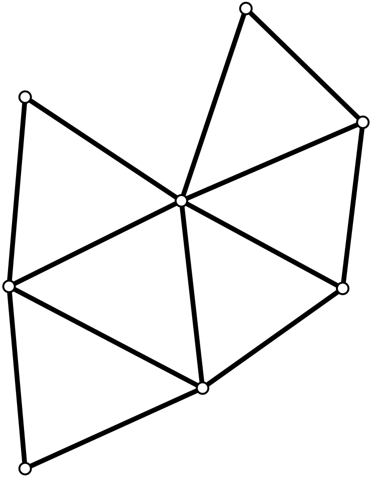


Does it glue?



Yes.

Does it glue?



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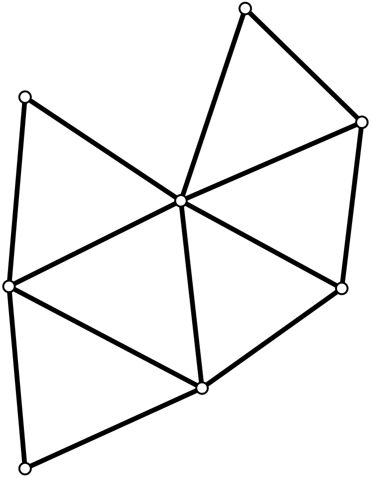
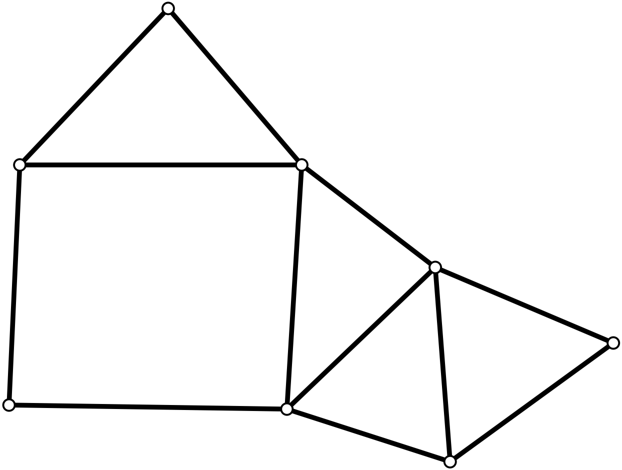


No.

Does it glue?



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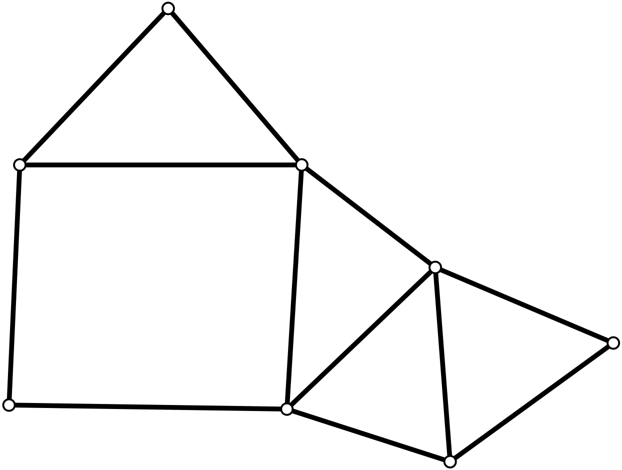
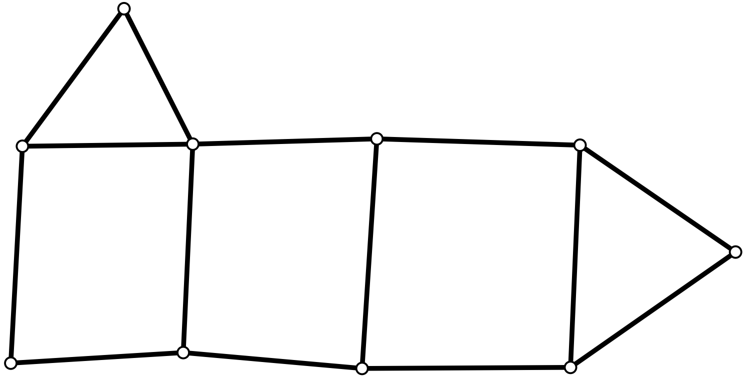
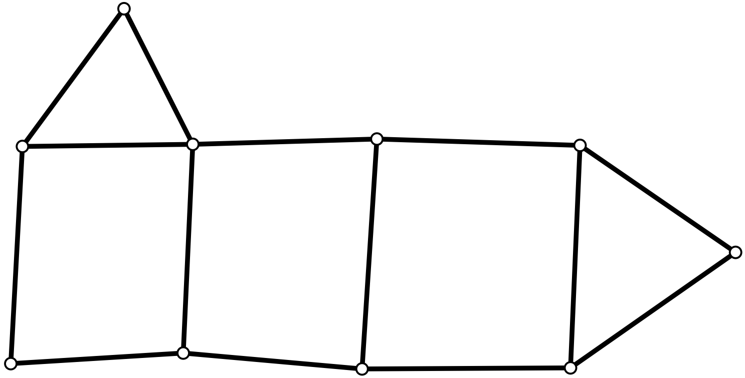


Yes.

Does it glue?



Does it glue?



No.

The most we can say....

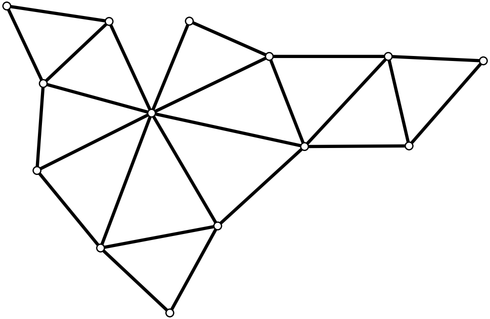
Conjecture

Suppose G is a maximal outerplanar graph with $|V(G)| = 2n$*.*
Then G admits a polyhedral gluing iff $\Delta(G) \leq n + 1$*.*

Proof by Induction?

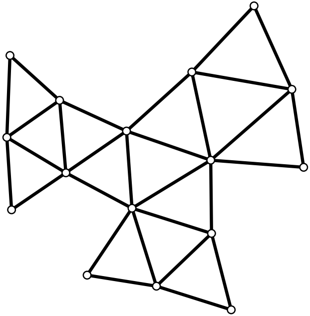
Glue adjacent outer edges, then contract an edge. (Use the chalkboard.)

Easy case



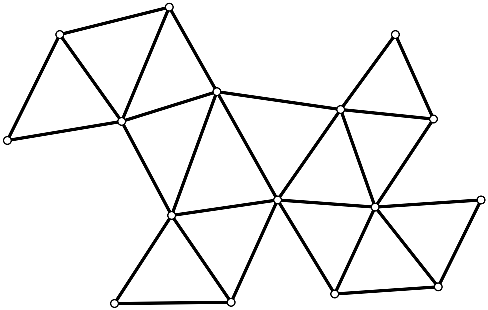
$$|V(G)| = 2n,\ \Delta(G) = n + 1.$$

Easy case



G has a “nice” degree-4 vertex.

Easy case

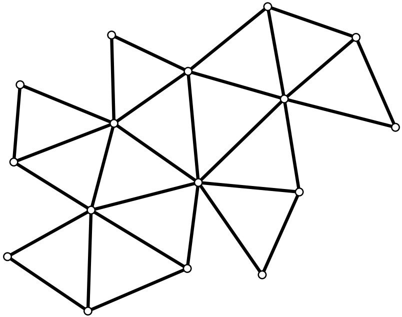


G has a degree-5 vertex below a (2,3) flap.

The Obstacle to Induction

A polyhedral gluing of a smaller graph may not extend to a polyhedral gluing of the larger graph. (Try it out!)

The hard case



Most look like this....

Another Problem

Characterize outerplanar graphs which have at most one polyhedral gluing.

The End