Choosability of Graphs with Bounded Order: Ohba's Conjecture and Beyond

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Joint work with

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Problem: Find the **choice number** $\mathrm{ch}(G)$:

$\mathrm{ch}(G)$ is the **minimum** $\boldsymbol{k}$ such that there is an acceptable colouring whenever $|L(v)| \geq k$ for all $v \in V(G)$.

$\chi(G)$ vs. $\mathrm{ch}(G)$

Example (Erdős, Rubin and Taylor 1979):

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Graph classes which are conjectured to satisfy $\mathrm{ch} = \chi$:

1. **line graphs** of multigraphs (many origins...).
2. **total graphs** of multigraphs (Borodin, Kostochka, Woodall 1997).
3. **squares of graphs** (Kostochka and Woodall 2001).
4. **claw-free graphs** (Gravier and Maffray 1997).

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Theorem (Kostochka, Stiebitz, Woodall 2011).
If $|V(G)| \leq 2\chi(G) + 1$ and $\alpha(G) \leq 5$, then $\mathrm{ch}(G) = \chi(G)$.

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If there is a matching in B_f that saturates V_f, then we are done.

Otherwise, by Hall's Theorem, there is a set $S \subseteq V_f$ such that $|N_{B_f}(S)| < |S|$.

Beyond Ohba's Conjecture

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However, perhaps we can still obtain a good bound on ch in terms of χ with a **less restrictive** bound on $|V(G)|$.

In particular, what is the best upper bound on $\mathrm{ch}(G)$ for graphs which satisfy $|V(G)| \leq 3\chi(G)$?

Graphs for which $|V(G)| \leq 3\chi(G)$

Theorem (Kierstead 2000).

$$\operatorname{ch}\left(K_{\underbrace{3,3,\ldots,3}_{k}}\right) = \left\lceil \frac{4k-1}{3} \right\rceil.$$

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Theorem (N., West, Wu, Zhu 2013). If $|V(G)| \leq 3\chi(G)$, then $\mathrm{ch}(G) \leq \left\lceil \frac{4\chi(G)-1}{3} \right\rceil$.

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Theorem (N., West, Wu, Zhu 2013). For every graph G,

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Note: This result implies Ohba's Conjecture, as well as the result on graphs for which $|V(G)| \leq 3\chi(G)$.

Directions for Future Work

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Question 2: What is the choice number of $K_{\underbrace{4, 4, \dots, 4}_{k}}$?

Lower bound: $\left\lfloor \frac{3k}{2} \right\rfloor$ due to Yang 2003.

Upper bound: $\left\lceil \frac{5k-1}{3} \right\rceil$ due to N., West, Wu, Zhu 2013.

Thanks

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Questions?