On the Maximum Orders of an Induced Forest, an Induced Tree, and a Stable Set

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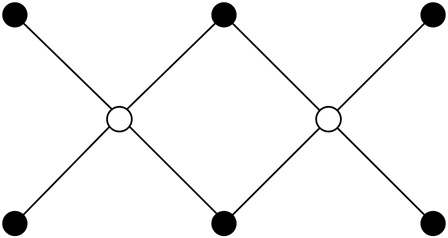
Linear Relations Between Invariants

- The Graffiti system (S. Fajtlowicz)
  1. $\mu(G) \leq \alpha(G)$ for any connected undirected graph G (Written on the Wall 2, 1986)
  2. $\mu(G) \leq \frac{\alpha_2(G)}{2}$ for any connected graph G (Written on the Wall 747, 1992)
- The AutoGraphiX system (G. Caporossi and P. Hansen)
  1. $E \geq 2\sqrt{m}$ (Caporossi, Cvetkovic, Gutman, Hansen)
  2. $E \geq \frac{4m}{n}$ (Caporossi, Cvetkovic, Gutman, Hansen)

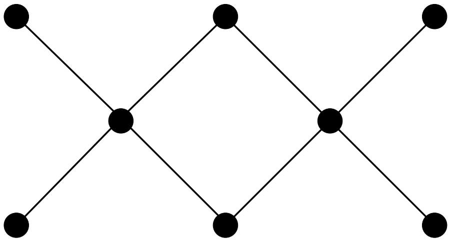
Some Graph Invariants: Maximum Size of a Stable Set



$$\alpha(G) = 6 > \frac{15}{7} = \mu(G)$$

$\alpha(G) \geq \mu(G)$ holds for all connected graphs (F. R. K. Chung, 1988)

Some Graph Invariants: Maximum Size of an Induced Bipartite Subgraph



$$\alpha(G) = 6 > 4 = \frac{\alpha_2(G)}{2} > \frac{15}{7} = \mu(G)$$

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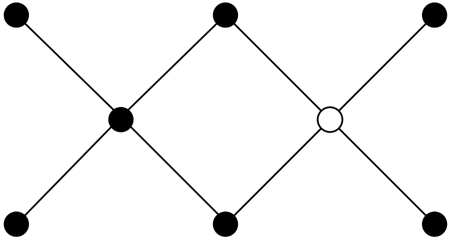
Some Graph Invariants: Maximum Size of an Induced Forest



$$\alpha(G) = 6 > 4 = \frac{\alpha_2(G)}{2} > \frac{7}{2} = \frac{F(G)}{2} > \frac{15}{7} = \mu(G)$$

$F(G)/2 \geq \mu(G)$ holds for all connected graphs (P. Hansen, A. Hertz, R. Kilani, O. M., D. Schindl, 2008)

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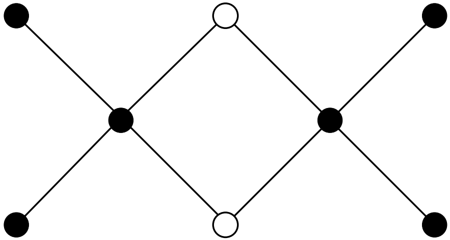
Some Graph Invariants: Maximum Size of an Induced Linear Forest



$$\alpha(G) = 6 > 4 = \frac{\alpha_2(G)}{2} > \frac{7}{2} = \frac{F(G)}{2} > 3 = \frac{LF(G)}{2} > \mu(G)$$

We conjecture that $LF(G)/2 \geq \mu(G)$ holds for all connected graphs.

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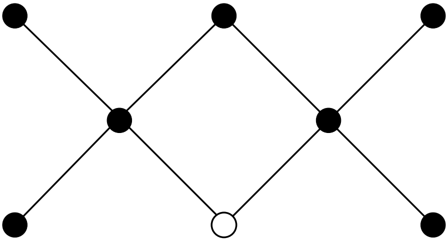
What About Induced Trees?



$T(G) = F(G) = 7$ for this graph

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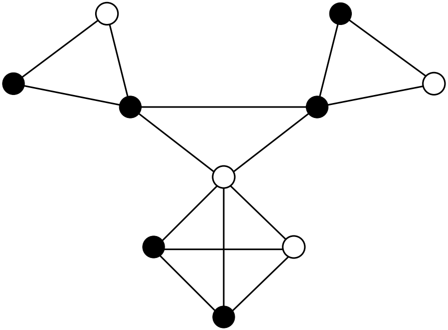
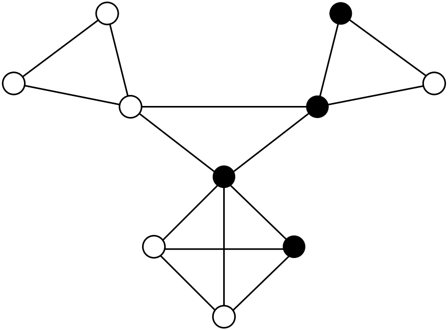
Another Example: an Induced Forest



$LF(G) = F(G)$ for this graph

Another Example: an Induced Tree



$$\frac{LF(G)}{2} = 3 > \frac{91}{45} = \mu(G) > 2 = \frac{T(G)}{2} \text{ for this graph}$$

Induced Forests and Induced Trees

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Main Result

For any connected graph G of order n and any induced forest F in G, there exists an induced tree in G whose order is at least equal to

$$\left\lceil \frac{|F|-2}{n+1-|F|} \right\rceil + 2.$$

Hence the relation

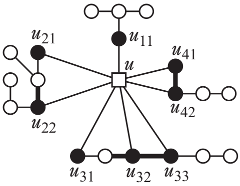
$$F(G) - T(G) \leq n - \left\lceil 2\sqrt{n-1} \right\rceil$$

holds for any connected graph G of order n. This bound is essentially tight.

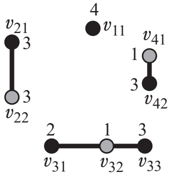
A Lemma on Induced Forests and Induced Trees

Lemma

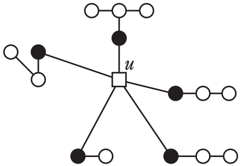
Let $G = (V, E)$ be a connected graph and assume that $F = V \backslash \{u\}$ induces a forest for some vertex u of G (i.e., V induces a quasi-forest). Then there exists an induced tree T in G containing u and whose order is at least $1 + \left\lceil \frac{|F|}{2} \right\rceil = 1 + \left\lceil \frac{|V|-1}{2} \right\rceil$.



1.a



1.b



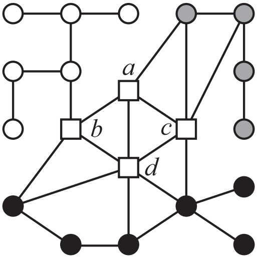
1.c

Attachment Sets

Let $G = (V, E)$ be a connected graph and F any induced forest in G. Let K denote $V \backslash F$ and w any vertex in F.

$$S_w = \{u \in K \mid \exists w' \text{ such that } uw' \in E \text{ and } w' \in C_w\}$$

S_w is called the attachment set of w.



A System of Linear Inequalities

Let x_S denote the number of vertices w in F verifying $S_w = S$.

$$\sum_{S \text{ contains } u} x_S \leq Z + 2, \qquad \forall u \in K$$

$$\sum_{S \cap P_{uv} = \{u\}} x_S + \sum_{S \cap P_{uv} = \{v\}} x_S \leq Z, \qquad \forall u, v \in K, u \neq v$$

$$\sum_S x_S = |F|$$

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Observations

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- If a constraint of the form

$$\sum_{S \text{ contains } u} x_S \leq Z + 2$$

  is tight, then G contains a quasi-forest of size $Z + 3$.
- If a constraint of the form

$$\sum_{S \cap P_{uv} = \{u\}} x_S + \sum_{S \cap P_{uv} = \{v\}} x_S \leq Z$$

  is tight, then G contains an induced subgraph of order $Z + 2$ that is the union of two quasi-forests and a shortest path between them.

We conclude that G contains an induced tree of order at least equal to $\lceil |Z|/2 \rceil + 2$.

An Upper Bound for $F(G) - T(G)$

If Z satisfies all of the above constraints, then we have

$$Z \geq 2(|F| - 2)/(n + 1 - |F|).$$

The graph G contains an induced tree of order at least equal to

$$\left\lceil \frac{|Z|}{2} \right\rceil + 2 = \left\lceil \frac{|F| - 2}{n + 1 - |F|} \right\rceil + 2.$$

We conclude that

$$F(G) - T(G) \leq n - \left\lceil 2\sqrt{n-1} \right\rceil$$

holds for any connected graph G.

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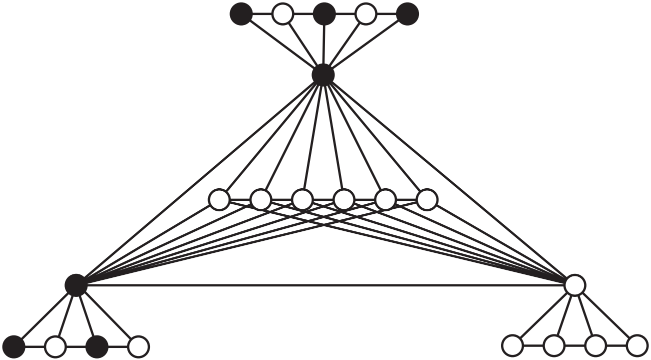
An Extremal Graph

This bound is tight in many cases. Here is an example for $n = 22$.



The Complete Picture

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Theorem

- Let G be a connected graph of order n, where n is of the form $a^2 + 1$ for some even positive integer $a \geq 4$. Then $F(G) - T(G) \leq n - \lceil 2\sqrt{n-1} \rceil - 1$ holds and the upper bound is tight.
- Let G be a connected graph of order n, where n is **not** of the form $a^2 + 1$ for some even positive integer $a \geq 4$. Then $F(G) - T(G) \leq n - \lceil 2\sqrt{n-1} \rceil$ holds and the upper bound is tight.

What about $\alpha(G) - T(G)$?

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Theorem

The relation

$$\alpha(G) - T(G) \leq n - \left\lceil 2\sqrt{2n} \right\rceil + 1$$

holds for any connected graph G of order n. The upper bound is tight.