How long does it take to catch a robber?

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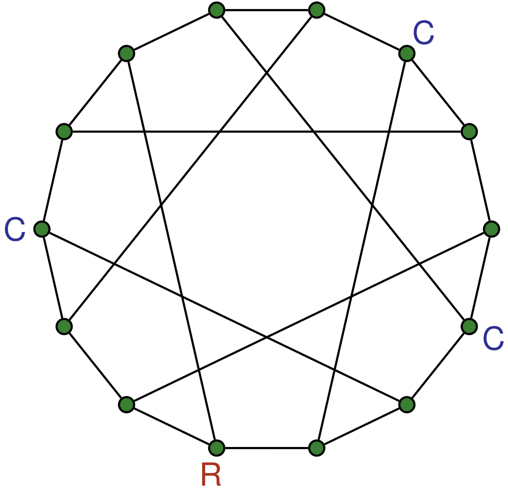
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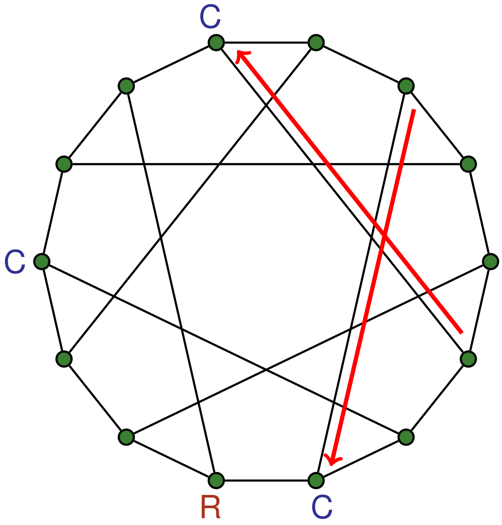
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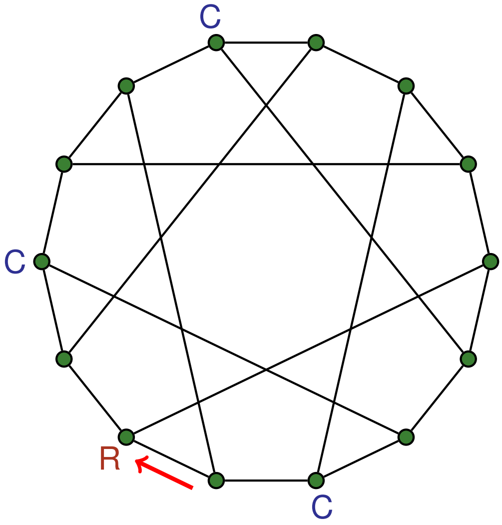
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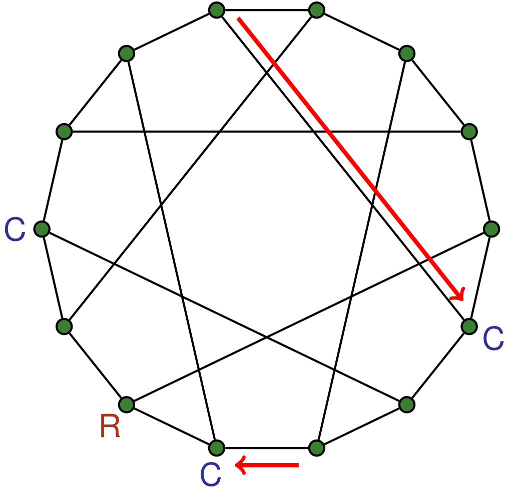
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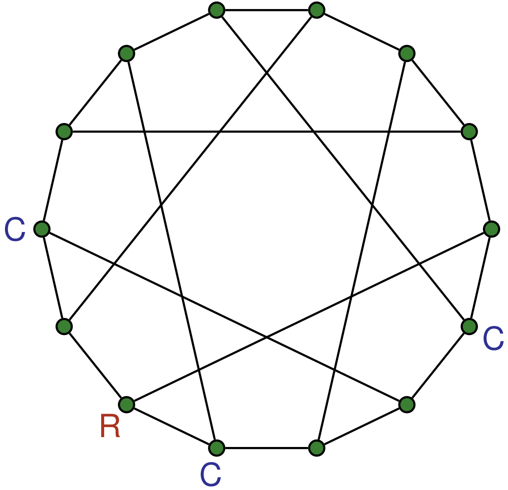
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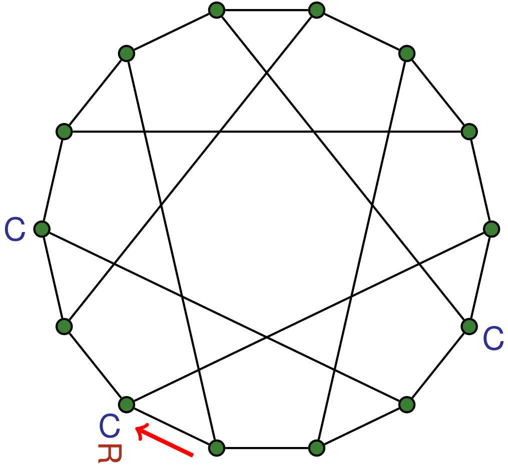
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Cops and Robbers

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- Perfect information: everyone always knows everything.
- Players occupy vertices of some graph.
- Cops choose initial positions first, followed by the robber.
- Players alternate turns: cops first, then robber. On each turn, players may either move to neighboring vertices, or stand still.
- If the robber ever occupies the same vertex as some cop, the cops win. The robber wins by evading capture forever.

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Definition

Given a graph G, the minimum number of cops needed to capture a robber on G is the cop number of G, denoted $c(G)$.

(Note: $|V(G)|$ cops always suffice.)

Capture time

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Let G be an n-vertex graph (with $n \geq 7$). If $c(G) = 1$, then $\mathrm{capt}(G) \leq n - 4$, and this is tight (even for planar graphs).

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Best general bound: $\mathrm{capt}(G) \leq n\binom{n+c(G)-2}{c(G)} + 1$. (Not a very good bound!)

A polynomial bound on $\mathrm{capt}(G)$ (not depending on $c(G)$) would imply that computing $c(G)$ is PSPACE-complete.

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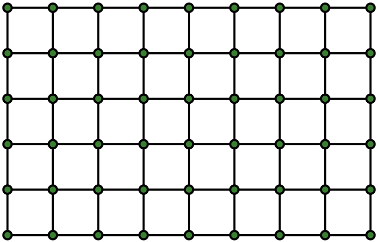

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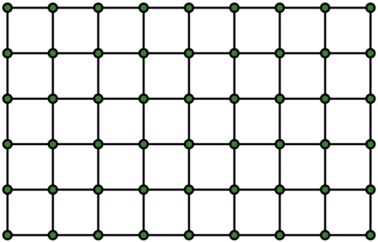

Natural next step: higher-dimensional grids.

Hypercubes

Our focus: the n-dimensional hypercube, Q_n.

- 2^n vertices – one for each n-bit binary string.
- Two vertices are joined by an edge if they differ in exactly one position.

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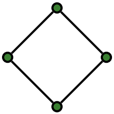

Q_2

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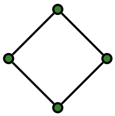
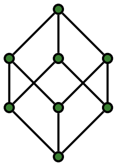


Q_2



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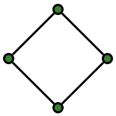
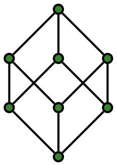


Q_2



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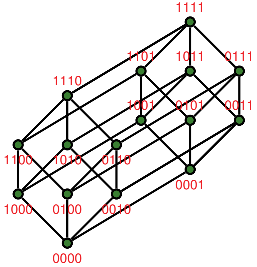

Q_4

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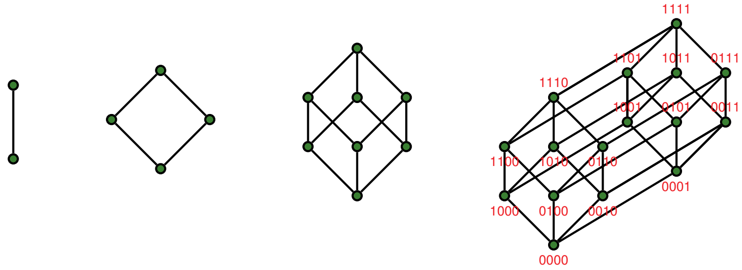

Q_1 Q_2 Q_3 Q_4

Theorem (Bonato, Gordinowicz, Kinnersley, Prałat '13+)

The n-dimensional hypercube has capture time $\Theta(n \ln n)$.

Capture time – upper bound

Upper bound: $\mathrm{capt}(Q_n) \leq (1 + o(1))n \log_2 n$.

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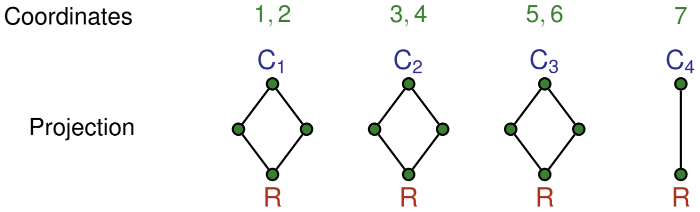

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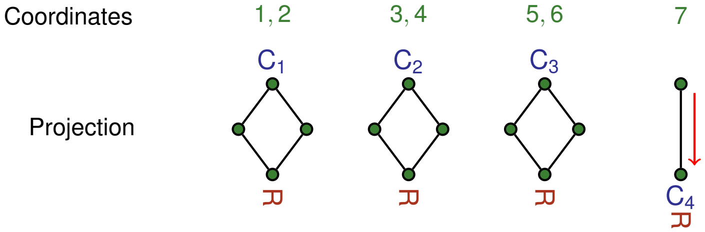

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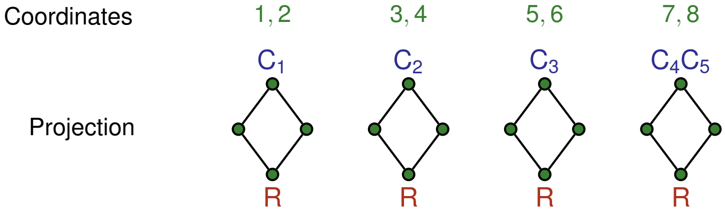

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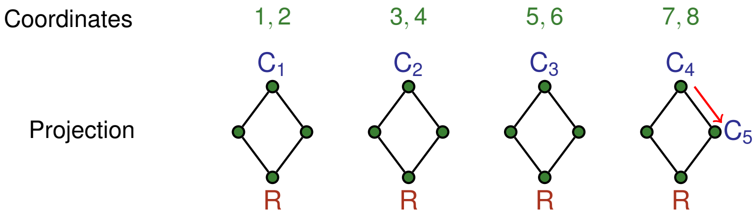

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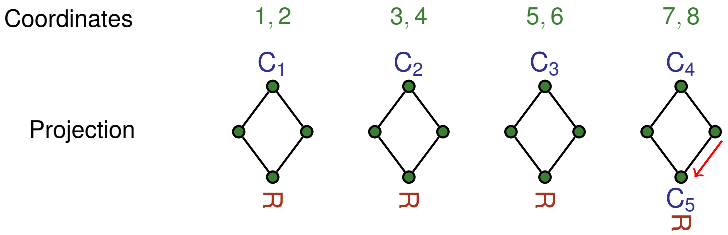

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  Each squad has multiple cops, all of whom move together as one.
- Each squad marks some coordinates active and the others inactive.
  Goal: capture the shadow of the robber in the active coords.

Capture time – upper bound

Initially, only two squads.

- The first sets coords $1, 2, \ldots, n/2$ active and the rest inactive.
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Eventually, each squad has just one cop – with two inactive coords.

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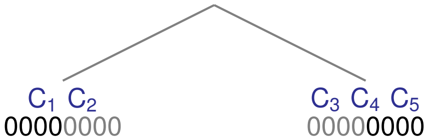
Example (Q_8)



Robber: 01110010

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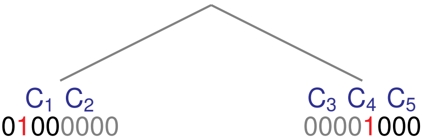

Robber: 01111010

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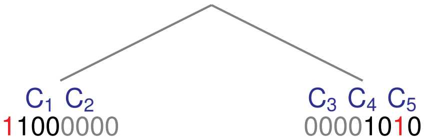

Robber: 11111010

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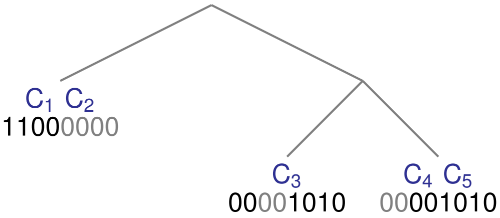

Robber: 11111010

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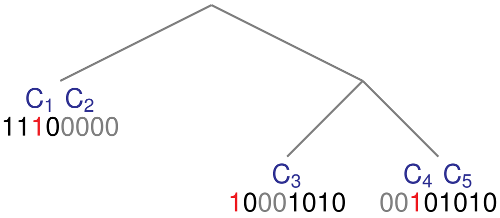

Robber: 11101010

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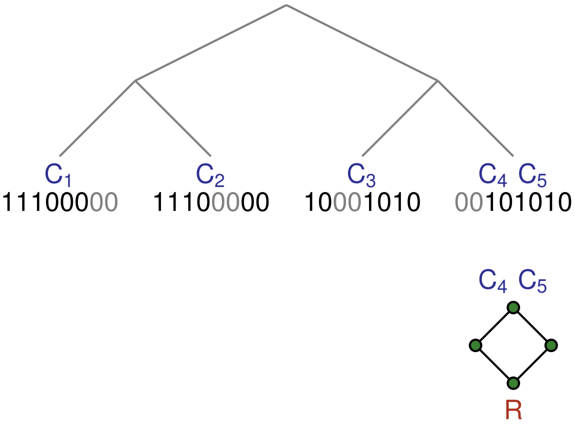

Robber: 11101010

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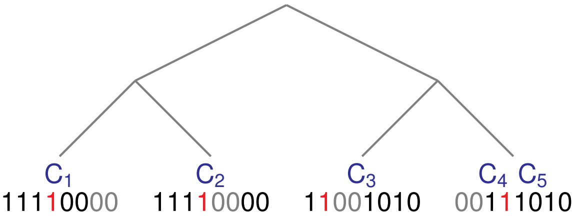

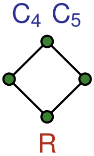

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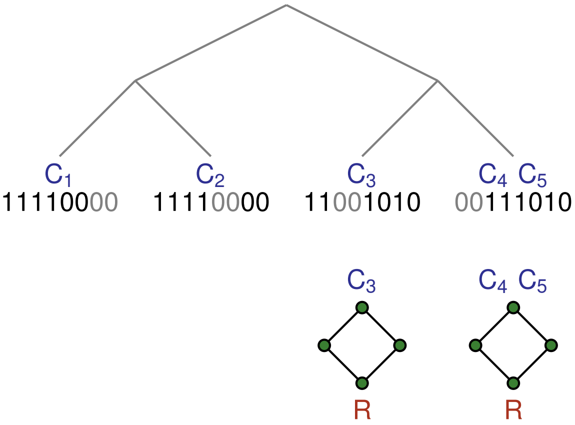

Robber: 11111010

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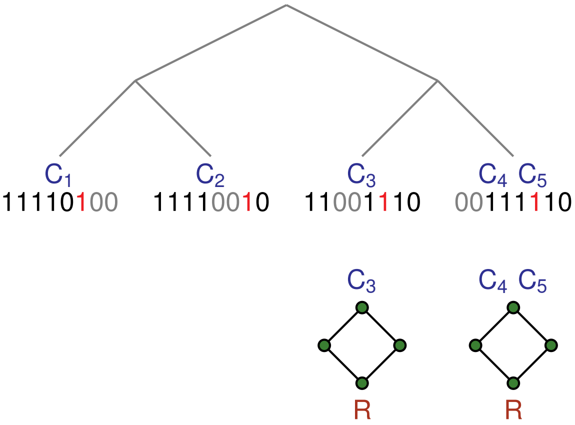

Robber: 11111110

Capture time – upper bound

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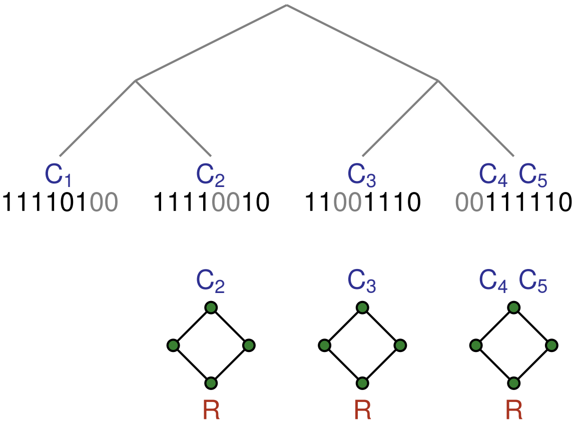

Robber: 11111110

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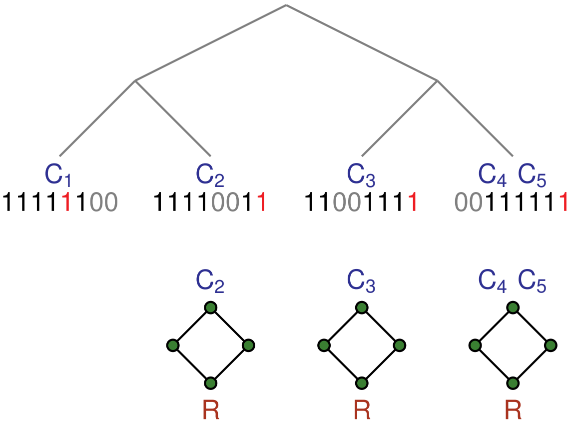

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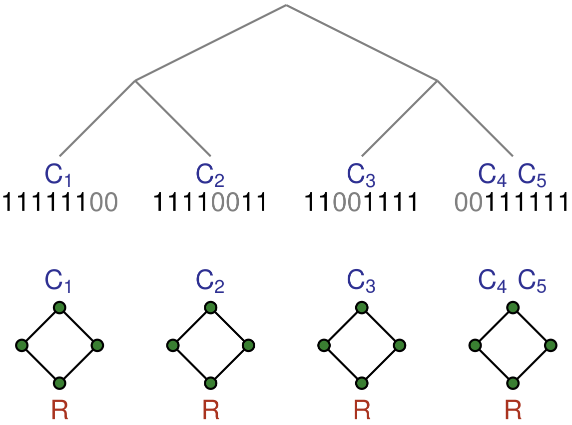

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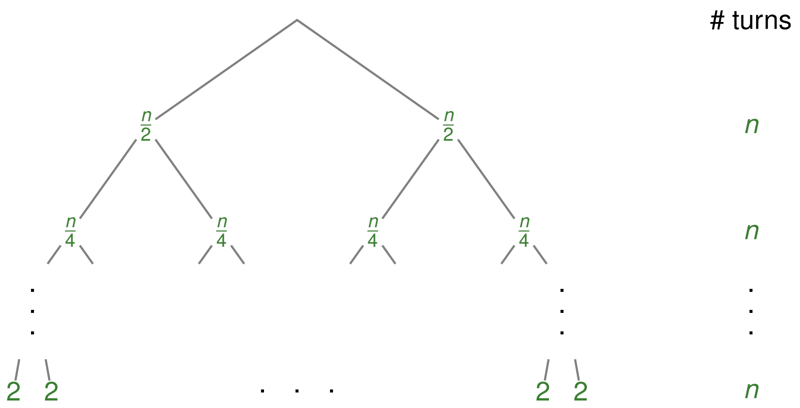
Capture time – upper bound

How long could this take?



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In fact, we can say a bit more:

Theorem (Bonato, Gordinowicz, Kinnersley, Prałat 13+)

For every integer n, and all trees $T_1, T_2 \ldots, T_n$, we have

$$\mathrm{capt}(T_1 \,\Box\, T_2 \,\Box\, \cdots \,\Box\, T_n) \leq \left(\sum_{i=0}^{n-1} \mathrm{rad}(T_i) \right) \lceil \log_2 n \rceil - \left\lfloor \frac{n-1}{2} \right\rfloor + 1.$$

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Analysis:

- Fix a strategy for the cops.
- We show that the probability of capture by one particular cop (within $\frac{1}{2} n \ln n$ rounds) is very small – so small that the probability of capture by any cop tends to 0.
- Consequently, some sequence of random choices for the robber "works" – so he has a deterministic strategy to survive.

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(Technicality: cop wants to stay at even distance. If at odd distance, sit still for one turn; after that, move greedily.)

Digression? Coupon collecting

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With k coupons already collected, probability of seeing a new type is $(m-k)/m$. Expected number of trials needed:

$$\frac{m}{m} + \frac{m}{m-1} + \frac{m}{m-2} + \ldots + \frac{m}{1} = m \sum_{i=1}^{m} \frac{1}{i},$$

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With d coupons uncollected, prob. of getting a new coupon: d/m

For cop/robber at distance d, prob. of getting closer: d/n

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Lemma

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$$\mathbf{Pr}\left[X < (1-\varepsilon)(m-1)\ln m\right] \leq \exp(-m^{-1+\varepsilon} m_0).$$

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Applying this to the game:

- Only $n/2$ "coupons" (since distance decreases by 2s)
- Initially, at least $n/8$ coupons uncollected (since robber starts at least distance $n/4$ away)

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Probability that a given cop captures the robber in under T rounds is at most

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Probability that any cop captures the robber in under T rounds is at most

$$\left\lceil \frac{n+1}{2} \right\rceil \exp\left(-\frac{(n/2)^{\varepsilon}}{4}\right) = o(1).$$

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For suitable choice of ε, we can make the argument work against any polynomial number of cops.

- How does the capture time change as we add more cops?
- Is the "drunk robber" still effective against many cops? (I think so.)
- Is the "drunk robber" still effective on larger grids? (Maybe not.)

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- When is drunkenness optimal? Almost optimal? How bad can it be? Work has been done on the drunk robber vs. one cop. How do the results compare?
- Is $\mathrm{capt}(G)$ always bounded by some polynomial in $|V(G)|$? (If yes, then computing the cop number is PSPACE-complete. If no, a constructive proof would still be interesting.)

Thanks

Thank you!