Extremal Hypergraphs for Packing and Covering

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Joint work with L. Narins and T. Szabó
Let $\mathcal{H}$ be a hypergraph. A packing or matching of $\mathcal{H}$ is a set of pairwise disjoint edges of $\mathcal{H}$.

The parameter $\nu(\mathcal{H})$ is defined to be the maximum size of a packing in $\mathcal{H}$. 

\[ \text{Packing} \]
Covering

A cover of the hypergraph $\mathcal{H}$ is a set of vertices $C$ of $\mathcal{H}$ such that every edge of $\mathcal{H}$ contains a vertex of $C$.

The parameter $\tau(\mathcal{H})$ is defined to be the minimum size of a cover of $\mathcal{H}$.
Comparing $\nu(\mathcal{H})$ and $\tau(\mathcal{H})$

For every hypergraph $\mathcal{H}$ we have $\nu(\mathcal{H}) \leq \tau(\mathcal{H})$.

For every $r$-uniform hypergraph $\mathcal{H}$ we have $\tau(\mathcal{H}) \leq r\nu(\mathcal{H})$. 
The upper bound $\tau(\mathcal{H}) \leq r\nu(\mathcal{H})$ is attained for certain hypergraphs, for example for the complete $r$-uniform hypergraph $\mathcal{K}_{rt+r-1}^r$ with $rt+r-1$ vertices, in which $\nu = t$ and $\tau = rt$. 
Ryser’s Conjecture

**Conjecture:** Let $\mathcal{H}$ be an $r$-partite $r$-uniform hypergraph. Then

$$\tau(\mathcal{H}) \leq (r - 1)\nu(\mathcal{H}).$$

This conjecture dates from the early 1970’s.
Results on Ryser’s Conjecture

- \( r = 2 \): This is König’s Theorem for bipartite graphs.

- \( r = 3 \): Known (proved by Aharoni, 2001)

- \( r = 4 \) and \( r = 5 \): Known for small values of \( \nu(\mathcal{H}) \), namely for \( \nu(\mathcal{H}) \leq 2 \) when \( r = 4 \) and for \( \nu(\mathcal{H}) = 1 \) when \( r = 5 \). (Tuza)

- Whenever \( r - 1 \) is a prime power: If true, the upper bound is best possible.
Here $\nu(\mathcal{H}) = 1$ and $\tau(\mathcal{H}) = r - 1$. 
On Ryser’s Conjecture for $r = 3$

Theorem (Aharoni 2001): Let $\mathcal{H}$ be a 3-partite 3-uniform hypergraph. Then

$$\tau(\mathcal{H}) \leq 2\nu(\mathcal{H}).$$

Proof: Uses topological connectedness of matching complexes of bipartite graphs.

Q: What is $\mathcal{H}$ like if it is a 3-partite 3-uniform hypergraph with $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$?
Extremal hypergraphs for Ryser’s Conjecture
Home base hypergraphs

[Diagram of hypergraphs with nodes and edges marked as 'F' and 'R']
Extremal hypergraphs for Ryser’s Conjecture

Theorem (PH, Narins, Szabó): Let \( \mathcal{H} \) be a 3-partite 3-uniform hypergraph with \( \tau(\mathcal{H}) = 2\nu(\mathcal{H}) \). Then \( \mathcal{H} \) is a home base hypergraph.
Some proof ingredients
The extremal result for Ryser’s conjecture for \( r = 3 \) initially follows Aharoni’s proof of the conjecture for \( r = 3 \), which uses Hall’s Theorem for hypergraphs together with König’s Theorem.

Hall’s Theorem: The bipartite graph \( G \) has a complete matching if and only if: For every subset \( S \subseteq A \), the neighbourhood \( \Gamma(S) \) is big enough.

Here big enough means \( |\Gamma(S)| \geq |S| \).
Hall’s Theorem for 3-uniform hypergraphs

Theorem (Aharoni, PH, 2000): The bipartite 3-uniform hypergraph $H$ has a complete packing if: For every subset $S \subseteq A$, the neighbourhood $\Gamma(S)$ has a matching of size at least $2(|S| - 1) + 1$. 

![Diagram of Hall's Theorem for 3-uniform hypergraphs](image)
Aharoni’s proof of Ryser for $r = 3$

Let $H$ be a 3-partite 3-uniform hypergraph. Let $\tau = \tau(H)$. Then by König’s Theorem, for every subset $S$ of $A$, the neighbourhood graph $\Gamma(S)$ has a matching of size at least $|S| - (|A| - \tau)$.

Then by a defect version of Hall’s Theorem for hypergraphs, we find that $H$ has a packing of size $\lfloor \tau/2 \rfloor$. 
Proof of Hall’s Theorem for hypergraphs

The proof has two main steps.

**Step 1:** The bipartite 3-uniform hypergraph $H$ has a complete packing if: For every subset $S \subseteq A$, the topological connectedness of the matching complex of the neighbourhood graph $\Gamma(S)$ is at least $|S| - 2$.

**Step 2:** If the graph $G$ has a matching of size at least $2(|S| - 1) + 1$ then the topological connectedness of the matching complex of $G$ is at least $|S| - 2$.

The matching complex of $G$ is the abstract simplicial complex with vertex set $E(G)$, whose simplices are the matchings in $G$. 
Topological connectedness

One way to describe topological connectedness of an abstract simplicial complex $\Sigma$, as it is used here:

We say $\Sigma$ is $k$-connected if for each $-1 \leq d \leq k$ and each triangulation $T$ of the boundary of a $(d+1)$-simplex, and each function $f$ that labels each point of $T$ with a point of $\Sigma$ such that the set of labels on each simplex of $T$ forms a simplex of $\Sigma$, the triangulation $T$ can be extended to a triangulation $T'$ of the whole $(d+1)$-simplex, and $f$ can be extended to a full labelling $f'$ of $T'$ with the same property.

Hall’s Theorem for hypergraphs uses this together with Sperner’s Lemma.

The topological connectedness of the matching complex of $G$ is not a monotone parameter.
Extremal hypergraphs for Ryser’s Conjecture

Two main parts are needed in understanding the extremal hypergraphs for Ryser’s Conjecture for \( r = 3 \).

**Part A:** Show that any bipartite graph \( G \) that has a matching of size \( 2k \) but whose matching complex has the smallest possible topological connectedness (namely \( k - 2 \)) has a very special structure.

**Part B:** Analyse how the edges of the neighbourhood graph \( G \) of \( A \) (which has this special structure) extend to \( A \).
Home base hypergraphs
Part B (one case)

There exists a subset $X$ of $C$ with $|Y| \leq |X|$, where $Y = \Gamma_G(X)$, such that for each $y \in Y$, if we erase the $(y, C \setminus X)$ edges of $G$, the topological connectedness of the matching complex goes up.
If for each $S \subset A$, the topological connectedness of the matching complex of $\Gamma(S)$ did not go down, then we find $H$ has a packing larger than $\nu(H)$.

So for some $S_y$, erasing the $(y, C \setminus X)$ edges causes the connectedness to decrease.

Properties of $S_y$:

• $|S_y| \geq |A| - 1$, which implies $S_y = A \setminus \{a\}$ for some $a \in A$,

• every maximum matching in $\Gamma(S)$ uses an edge of $(y, C \setminus X)$. 
Removing the vertices in $Y$ and $Z$ causes $\nu$ to decrease by $|Y|$ and $\tau$ to decrease by $2|Y|$. Then we may use induction.