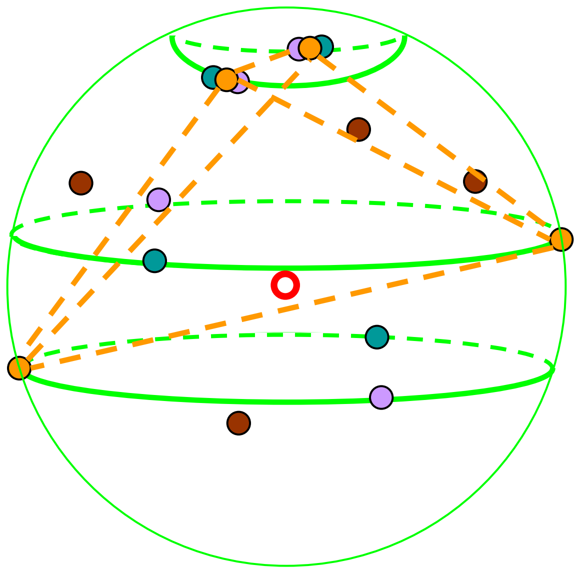
# *Combinatorial, computational, and geometric approaches to colourful simplical depth*



**Antoine Deza** (McMaster)

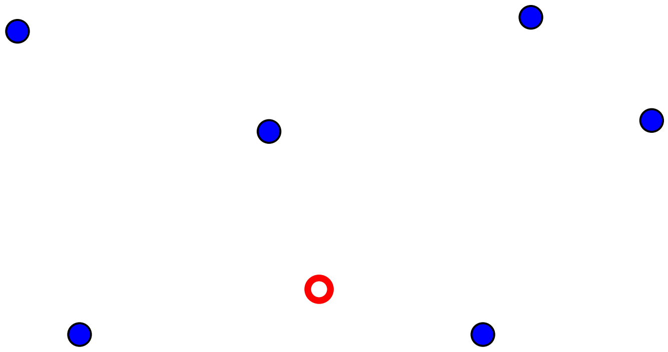
based on joint work with

**Frédéric Meunier** (ENPC)

**Pauline Sarrabezolles** (ENPC)

McMaster University
Advanced Optimization Laboratory

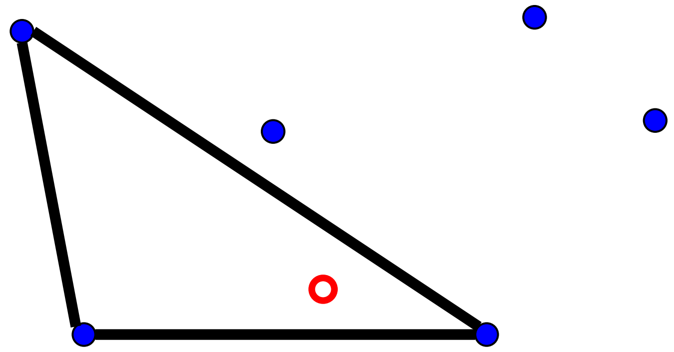
# *Carathéodory Theorem*



Given a set $S$ of $n$ points in dimension $d$, then there exists an open simplex generated by points in $S$ containing $p$

$S$, $p$ *general position*

# *Carathéodory Theorem*



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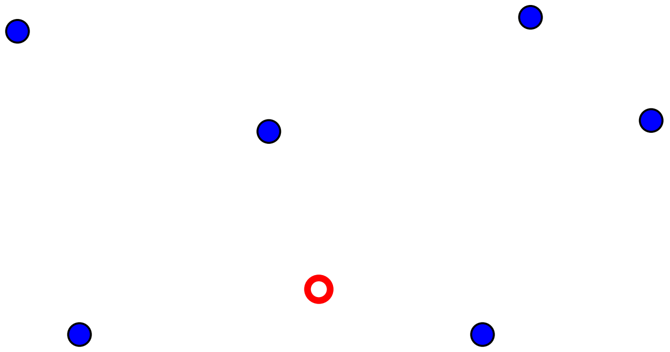
# *Simplicial Depth*

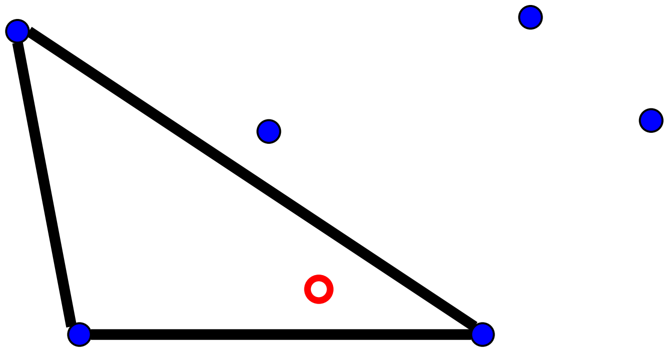


Given a set *S* of *n* points in dimension *d*, the *simplicial depth* of *p* is the number of open simplices generated by points in *S* containing *p* [Liu 1990]

*S*, *p* *general position*

# *Simplicial Depth*

$depth_S(p)=1$



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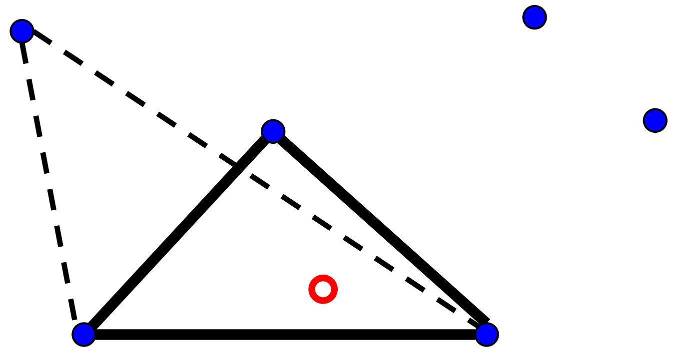


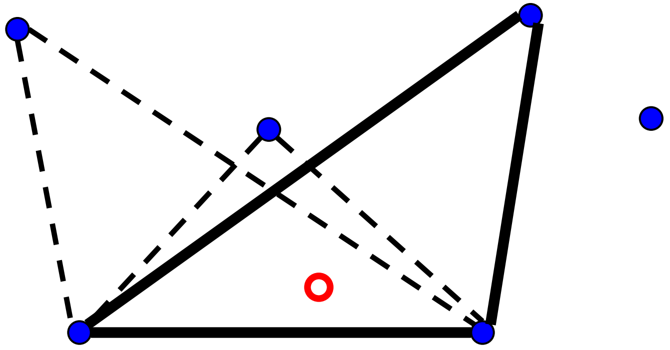
$depth_S(p)=2$

Given a set $S$ of $n$ points in dimension $d$, the *simplicial depth* of $p$ is the number of open simplices generated by points in $S$ containing $p$ [Liu 1990]

*$S$, $p$ general position*

# Simplicial Depth

$depth_S(p)=3$



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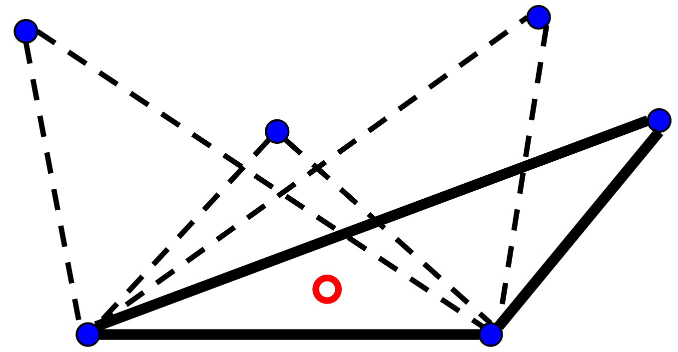
*$S$, $p$ general position*

# Simplicial Depth

$depth_S(p)=4$



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# *Simplicial Depth*

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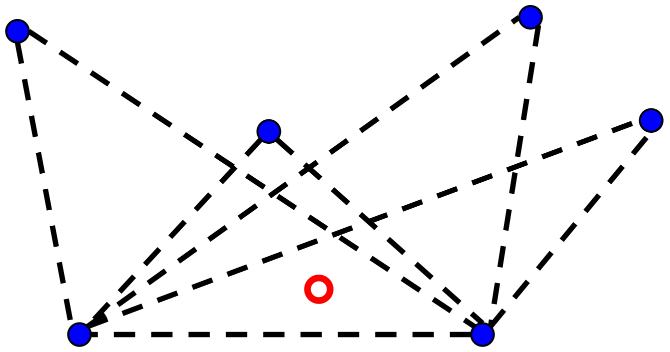

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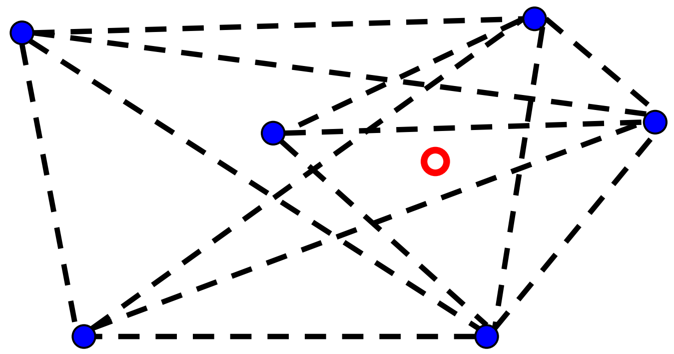
*$S$, $p$ general position*

# Simplicial Depth

$depth_S(p)=9$



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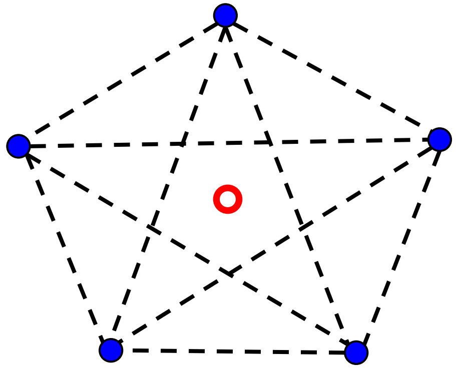
*$S$, $p$ general position*

# *Deepest Point in Dimension 2*

Deepest point bounds in dimension 2 [Kárteszi 1955], [Boros, Füredi 1984]

$$\frac{n^3}{27} + O(n^2) \le \max_p depth_S(p) \le \frac{n^3}{24} + O(n^2)$$

and [Bukh, Matoušek, Nivasch 2010]



$depth_S(p) = 5$

$S$, $p$ *general position*

# *Deepest Point in Dimension d*

Deepest point bounds in dimension *d* [Bárány 1982]

$$\frac{1}{(d+1)^{d+1}}\binom{n}{d+1}+O(n^d)\le \max_{p} depth_S(p)\le \frac{1}{2^d(d+1)!}n^{d+1}+O(n^d)$$

*S*, *p* *general position*

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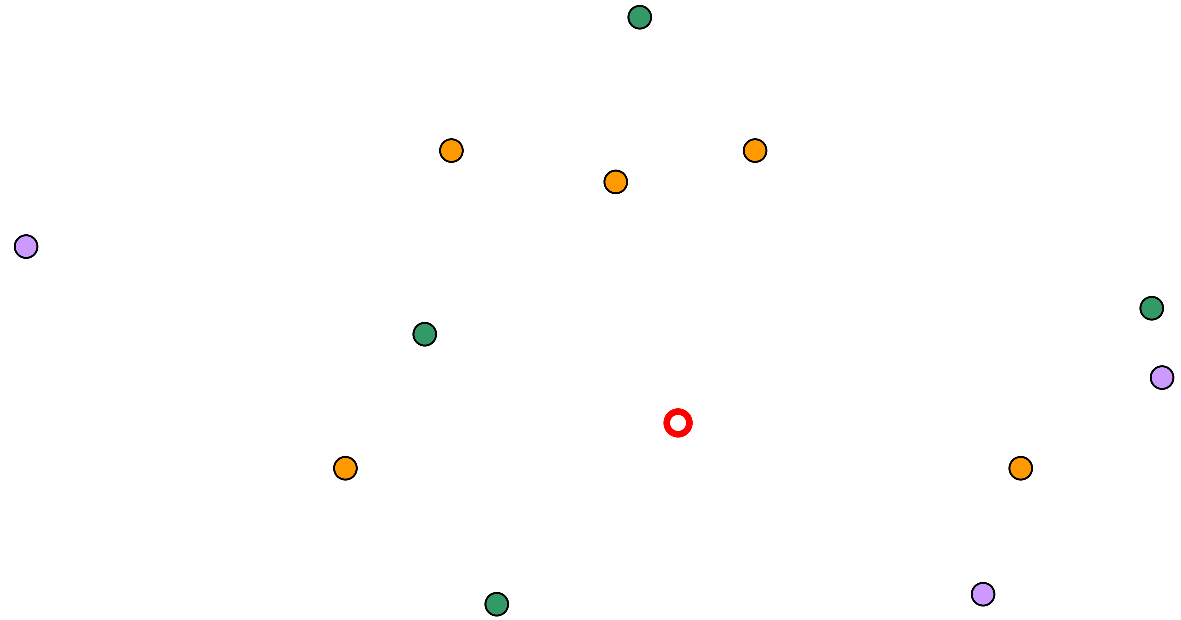
- tight upper bound
- lower bound uses Colourful Carathéodory theorem

… breakthrough [Gromov 2010] & further improvements

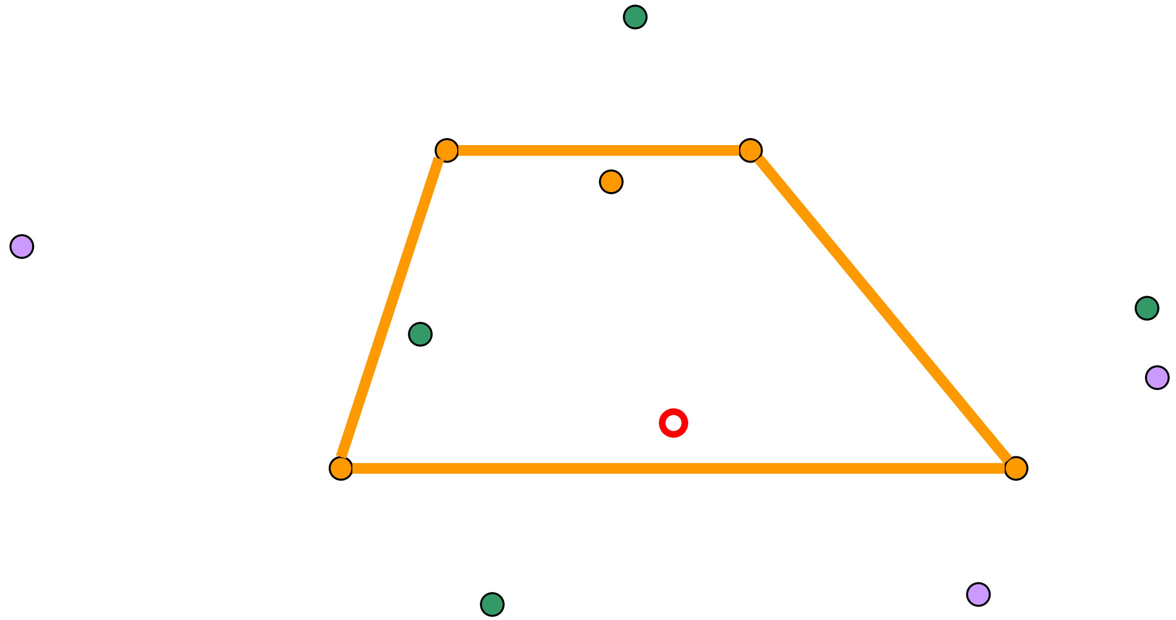
*$S$, $p$ general position*

# Colourful Carathéodory Theorem



Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension $d$, and $p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \ldots \cap \text{conv}(S_{d+1})$, there exists a colourful simplex containing $p$ [Bárány 1982]

# *Colourful Carathéodory Theorem*



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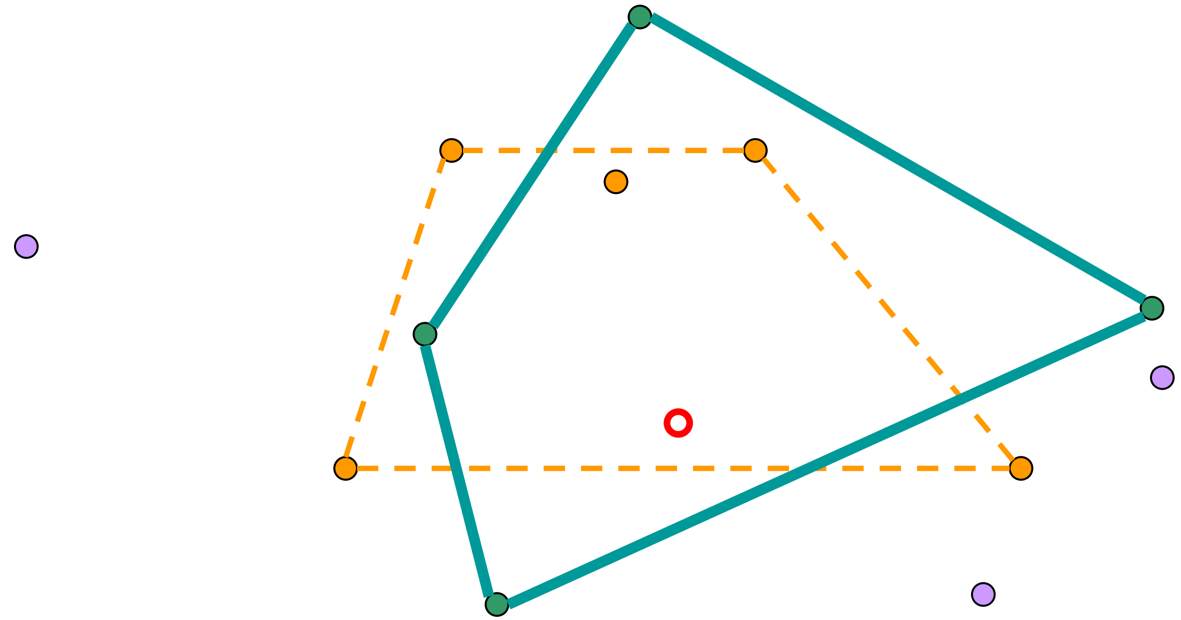

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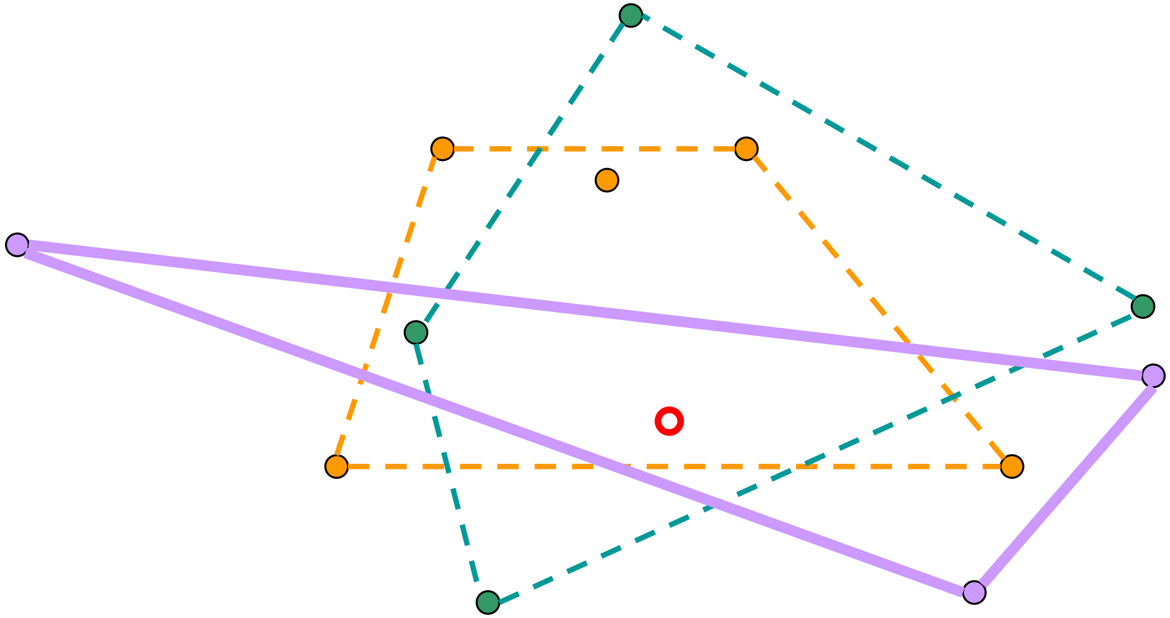

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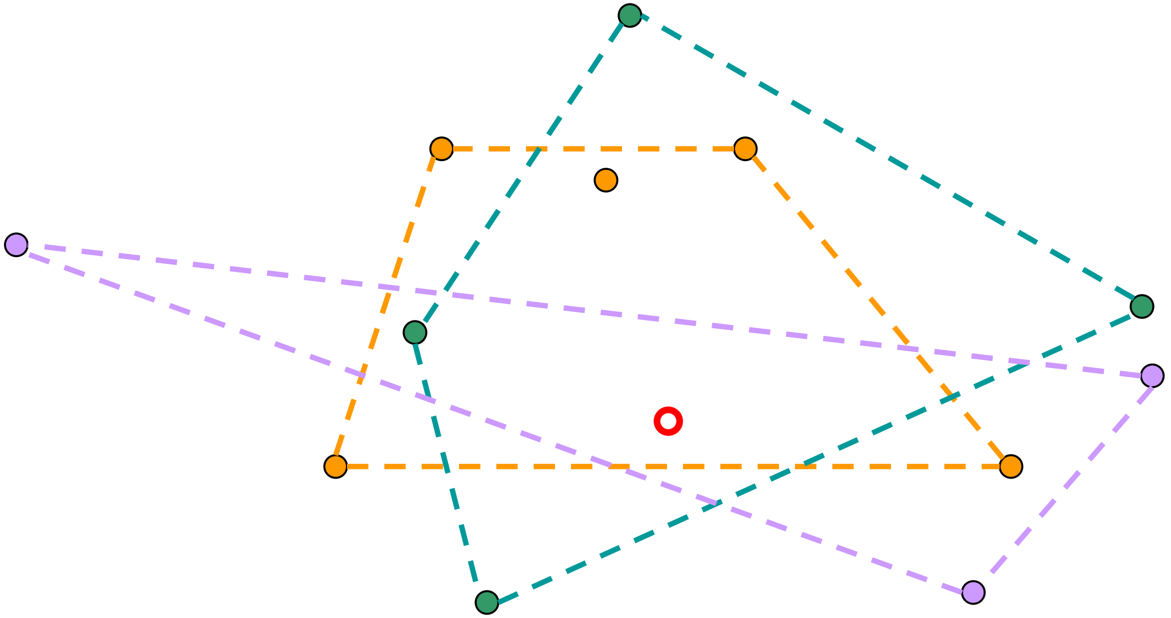

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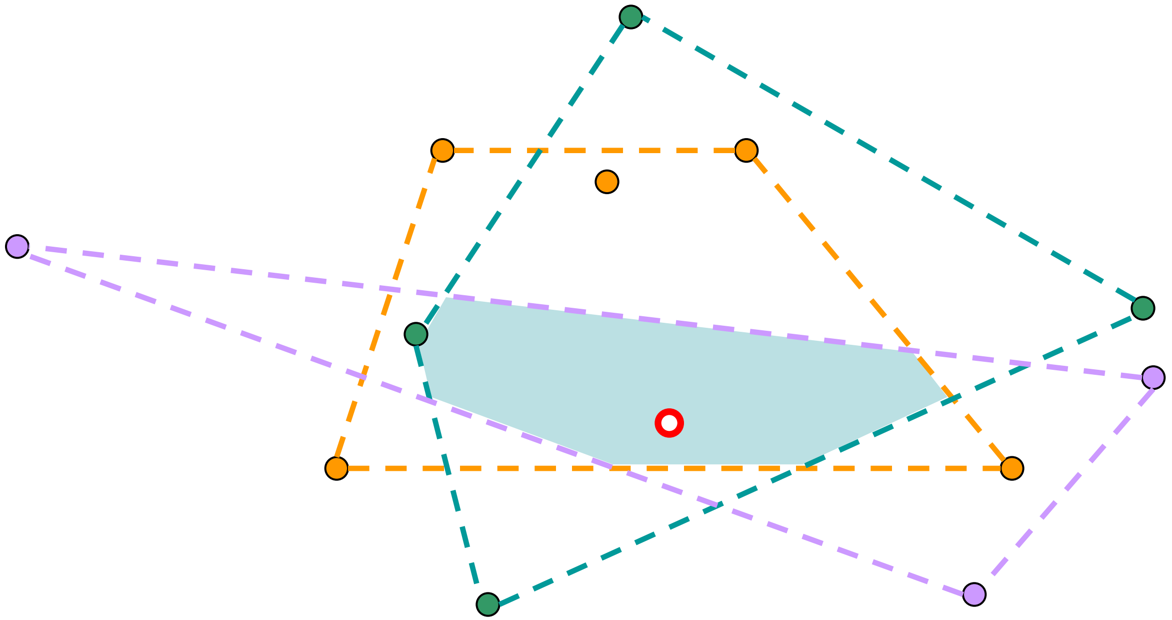

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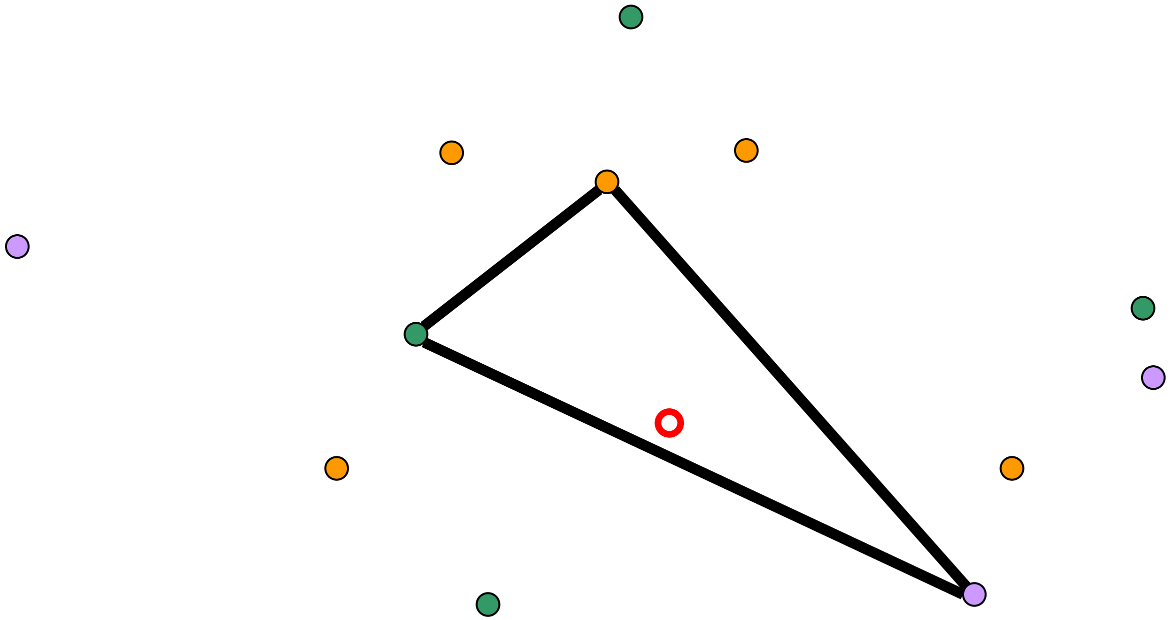

$depth_{\boldsymbol{S}}(p)=1$

Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension $d$, the *colourful simplicial depth* of $p$ is the number of open colourful simplexes generated by points in $\boldsymbol{S}$ containing $p$

*$\boldsymbol{S}$, $p$ general position*

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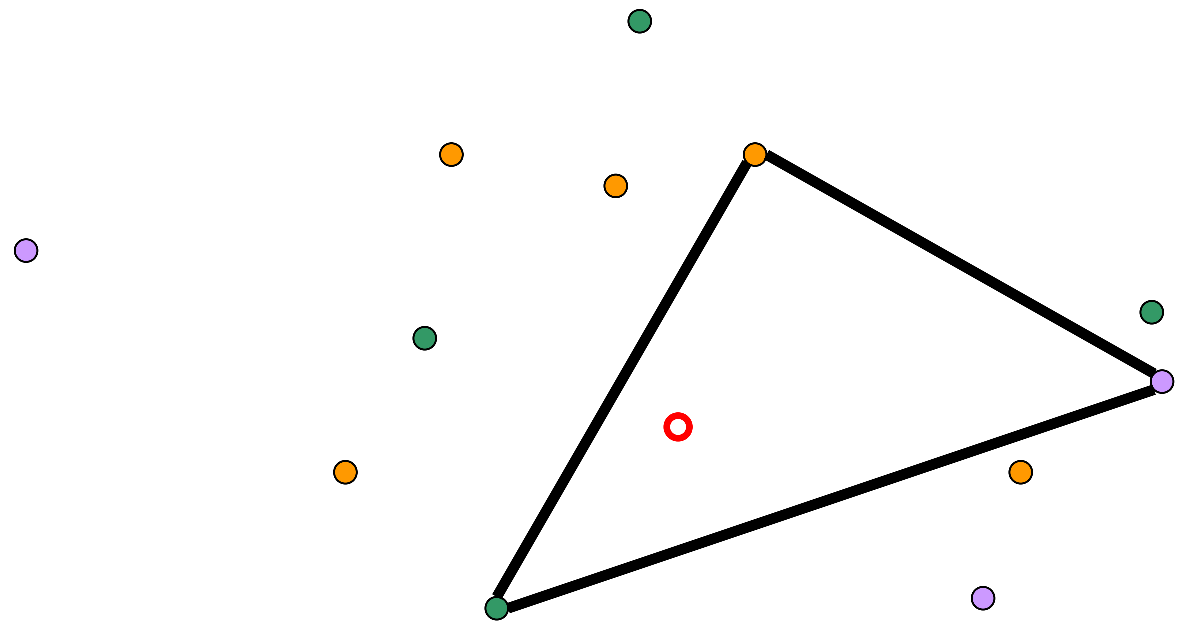
# *Colourful Simplicial Depth*

$depth_{\boldsymbol{S}}(p) = 2$



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***S***, *p general position*

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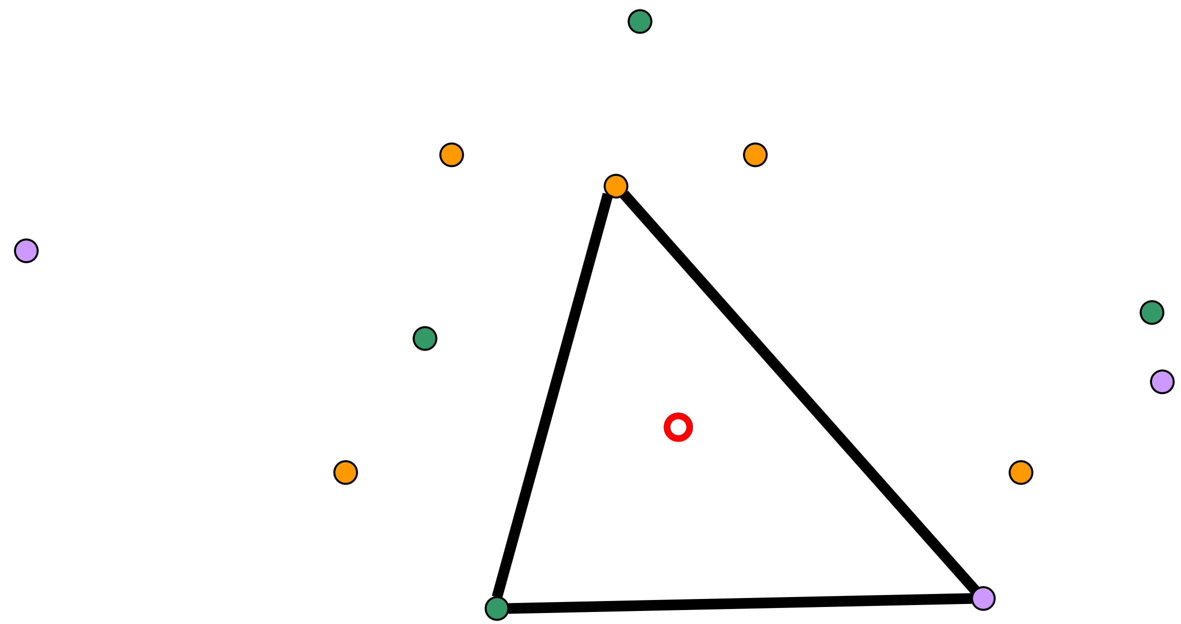
# *Colourful Simplicial Depth*

$depth_{\boldsymbol{S}}(p) = 3$



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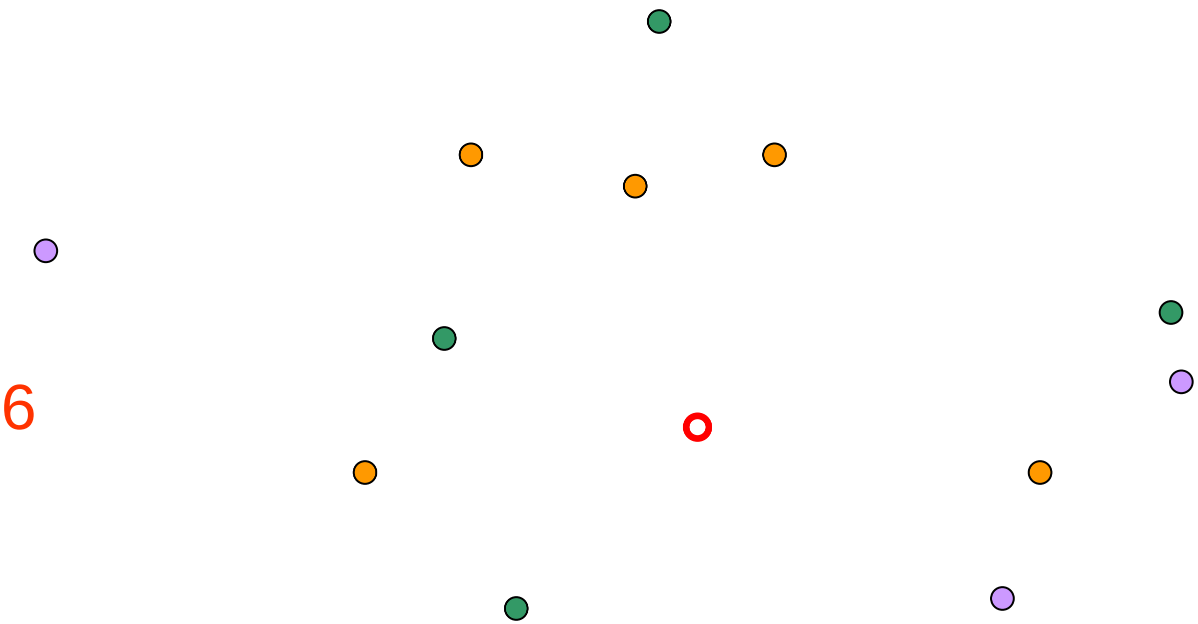

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# Deepest Point in Dimension d

Deepest point bounds in dimension $d$ [Bárány 1982]

$$\frac{\mu(d)}{(d+1)^{d+1}}\binom{n}{d+1}+O(n^{d})\le \max_{p} depth_{S}(p)\le \frac{1}{2^{d}(d+1)!}n^{d+1}+O(n^{d})$$

with $\mu(d)=\min_{S,p} depth_{S}(p)$

[Bárány 1982]: $\mu(d)\ge 1$

… breakthrough [Gromov 2010] & further improvements

*S, p general position*

# Deepest Point in Dimension d

$$\max_{p} depth_S(p) \geq c_d \binom{n}{d+1}$$

[Bárány 1982] $c_d \geq \frac{d+1}{(d+1)^{(d+1)}}$

[Wagner 2003] $c_d \geq \frac{d^2+1}{(d+1)^{(d+1)}}$

[Gromov 2010] $c_d \geq \frac{2d}{(d+1)!(d+1)}$

simpler proofs: [Karazev 2012], [Matoušek, Wagner 2012]

*d*=2: [Boros, Füredi 1984], [Bukh 2006]

*d*=3: [Král, Mach, Sereni 2012]

*S*, *p* *general position*

# *Colourful Research Directions*

- Generalize the sufficient condition of Bárány for the existence of a colourful simplex
- Improve lower bound for $\mu(d) = \min_{S,p} depth_S(p)$
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***S***, *p general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \geq d+1$

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# Colourful Carathéodory Theorems

[Bárány 1982] Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension $d$, and $p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \ldots \cap \text{conv}(S_{d+1})$, then there exists a colourful simplex containing $p$

[Holmsen, Pach, Tverberg 2008] and [Arocha, Bárány, Bracho, Fabila, Montejano 2009] Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension $d$, and $p \in \text{conv}(S_\text{i} \cup S_\text{j})$ for $1 \leq i < j \leq d+1$, then there exists a colourful simplex containing $p$

[Meunier, D. 2013] Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension $d$, if for $1 \leq i < j \leq d+1$ there exists $k \neq i$, $k \neq j$, such that for all $x_k \in S_k$ the ray $[x_k\, p)$ intersects $\text{conv}(S_\text{i} \cup S_\text{j})$ in a point distinct from $x_k$, then there exists a colourful simplex containing $p$

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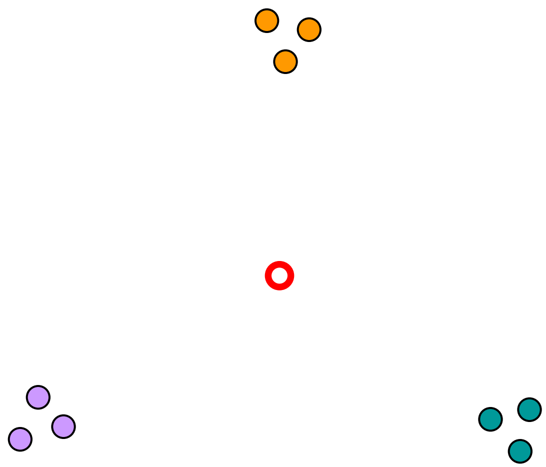

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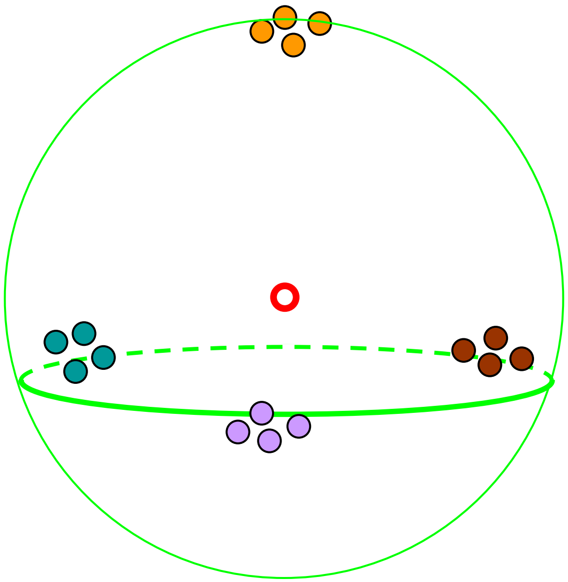
# *Colourful Carathéodory Theorems*



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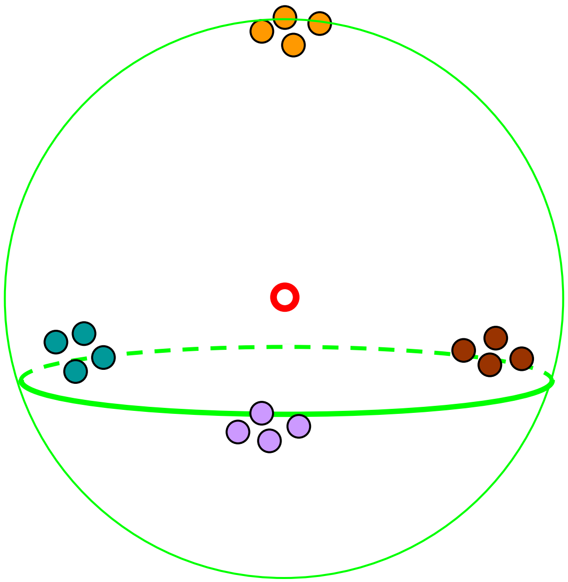

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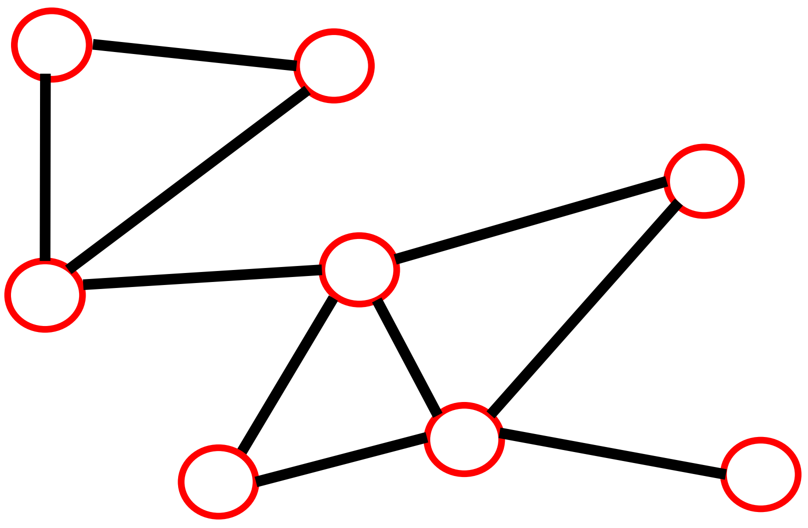
❖ *further generalization in dimension 2*

# *Given One, Get Another One*



In a graph, if there is a vertex with an odd degree…

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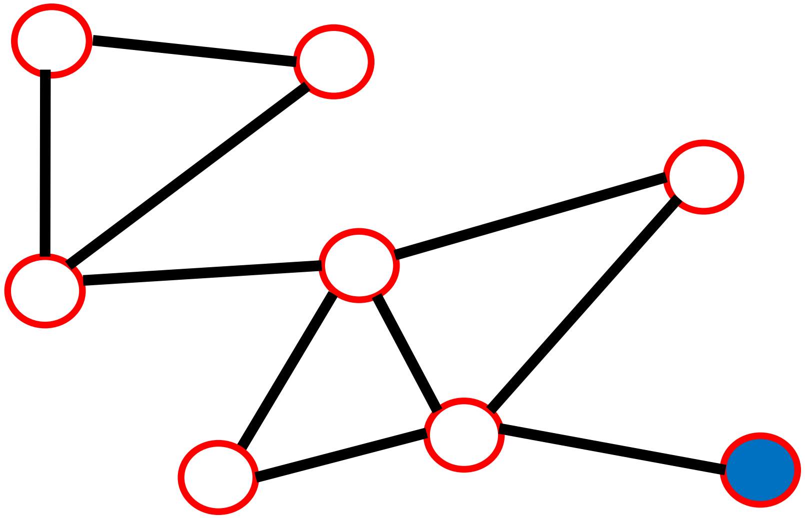

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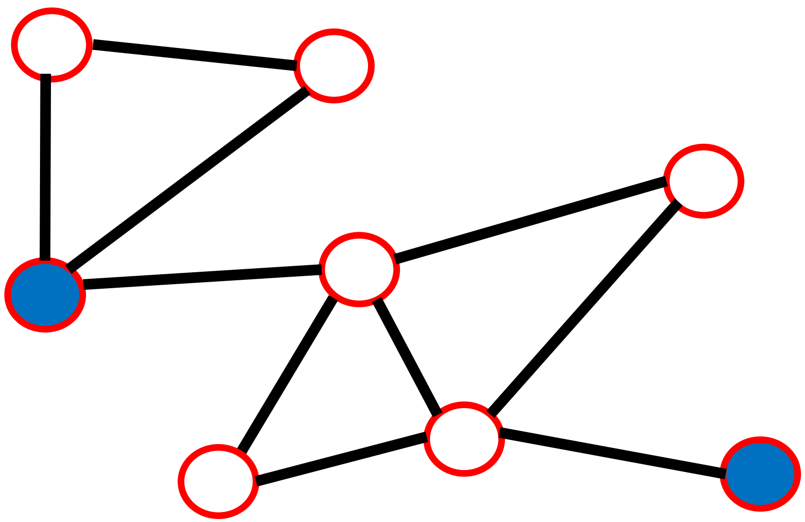

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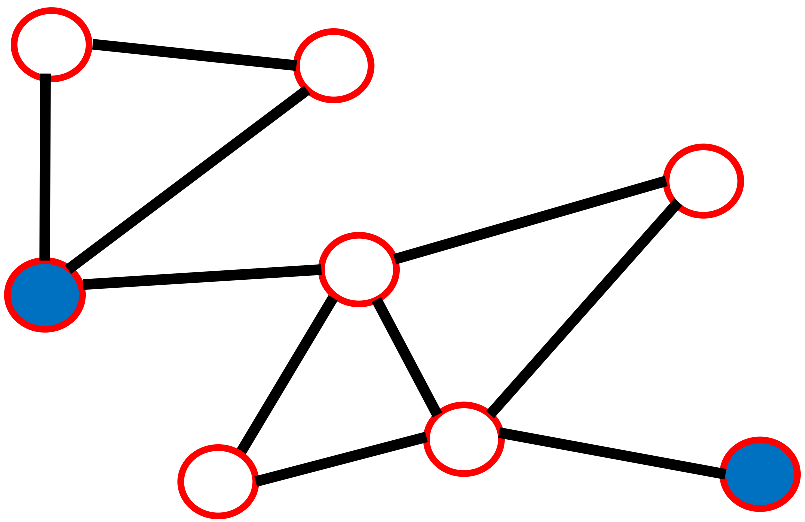

In a graph, if there is a vertex with an ***odd degree***… then there is ***another one***

- *Duoid / oik* room partitioning Todd 1974, Edmonds 2009]

(*Exchange algorithm*: generalization of Lemke-Howson for finding a Nash equilibrium for a 2 players game)

- *Polynomial Parity Argument* PPA(D) [Papadimitriou 1994]

(Hamiltonian circuit in a cubic graph, Borsuk-Ulam, …)

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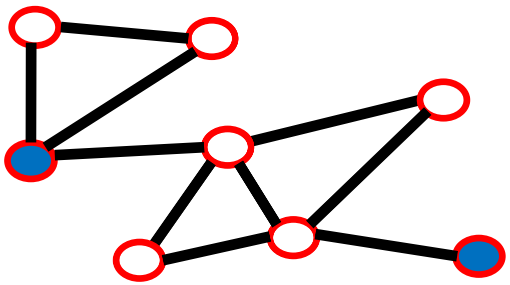

[Meunier, D. 2013] Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension $d$ with $|S_i|=2$, if there is a colourful simplex containing $p$ then there is another one

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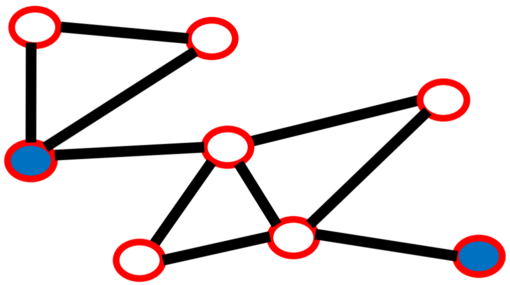

[Meunier, D. 2013]  Any condition implying the existence of a colourful simplex containing $p$ actually implies that the number of such simplices is at least $d$+1

***S***, $p$ *general position* and $|S_1|,|S_2|,\ldots,|S_{d+1}| \geq d+1$

# Colourful Research Directions

- Generalize the sufficient condition of Bárány for the existence of a colourful simplex
- Improve lower bound for $\mu(d) = \min_{S,p} depth_S(p)$
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$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S_2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$

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# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

[Bárány 1982]

$$1 \le \mu(d)$$

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# *Colourful Simplicial Depth Bounds*

$$\mu(d) = \min_{S,p} depth_S(p)$$

[Bárány 1982]

$$d+1 \le \mu(d)$$

[D.,Huang,Stephen,Terlaky 2006]

$$2d \le \mu(d) \le d^2+1$$

$$\mu(d) \text{ even for odd } d$$

$$\max_p depth_S(p) \ge \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \ldots \cap \text{conv}(S_{d+1})$

***S***, $p$ *general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \ge d+1$

# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

[Bárány 1982] $d+1 \le \mu(d)$

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$\mu(d)$ even for odd $d$

[Bárány, Matoušek 2007] $\max(3d, \frac{d^2+d)}{5}) \le \mu(d)$ for $d \ge 3$

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$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

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# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

$$\mu(1) = 2 \qquad \mu(2) = 5 \qquad \mu(3) = 10$$

$$\left\lceil \frac{(d+1)^2}{2} \right\rceil \le \mu(d) \le d^2 + 1 \quad \text{for } d \ge 4$$

$\mu(d)$ even for odd $d$

conjecture: $\mu(d) = d^2 + 1$

$$\max_p depth_S(p) \ge \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S_2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$

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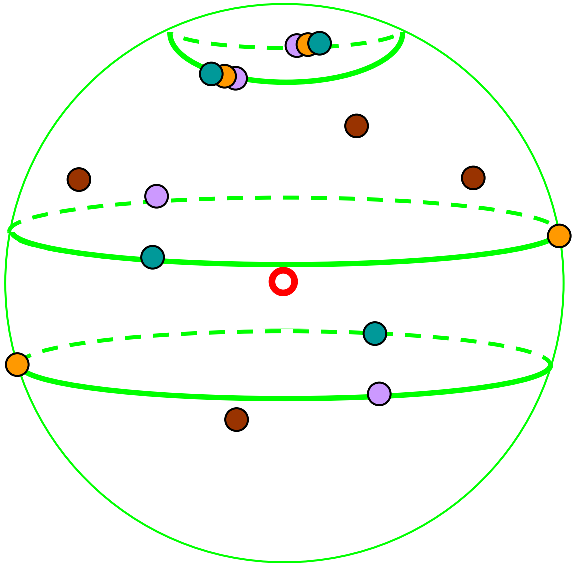

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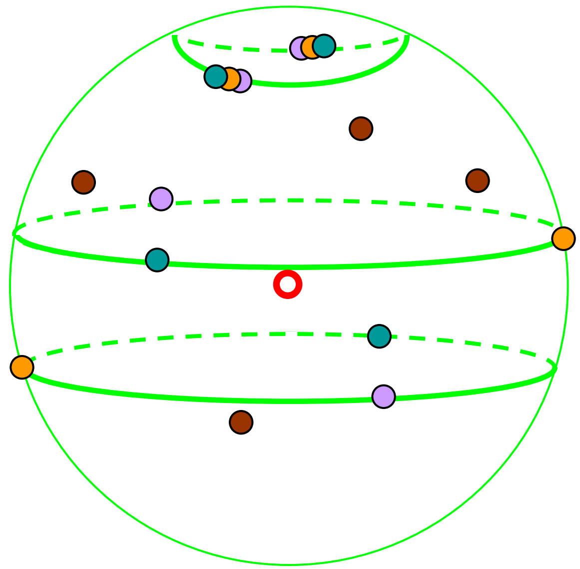
# *Colourful Simplicial Depth Bounds*

$$\mu(d) = \min_{S,p} depth_S(p)$$



$$9 \le \mu(3) \le 10$$

$p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S_2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$

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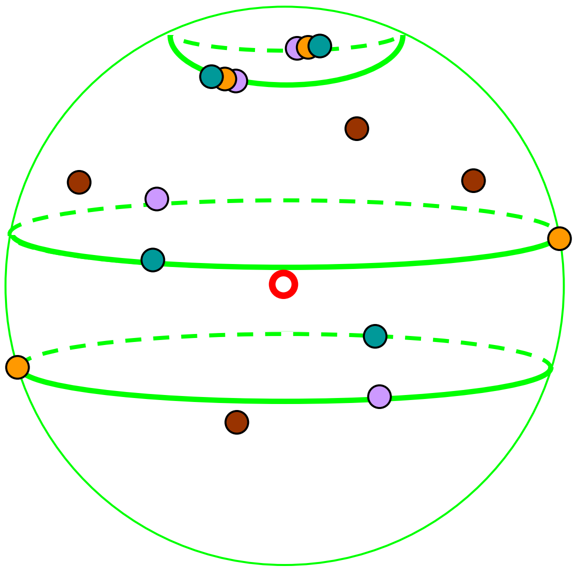

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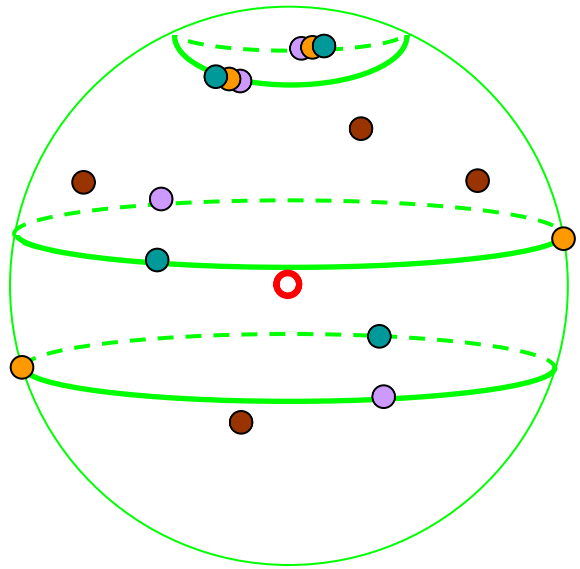

$$\mu(3) = 10$$

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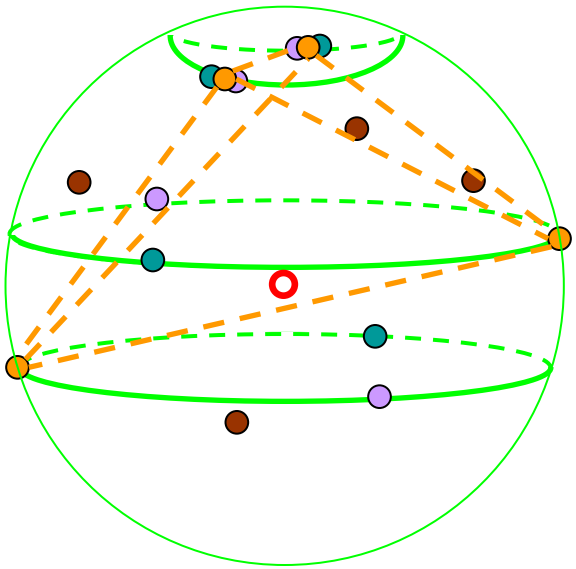

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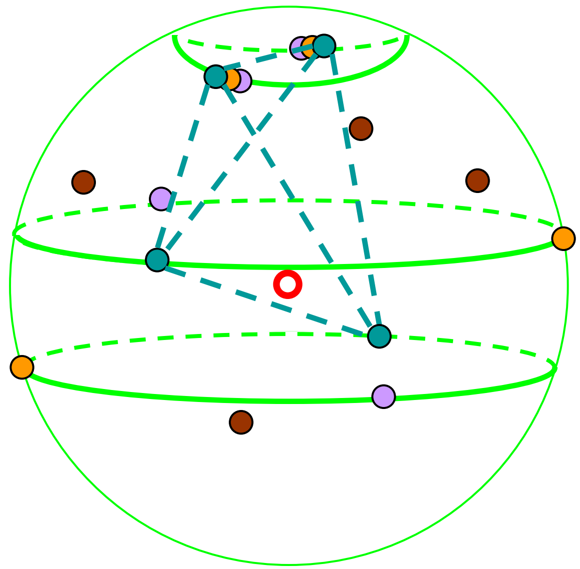

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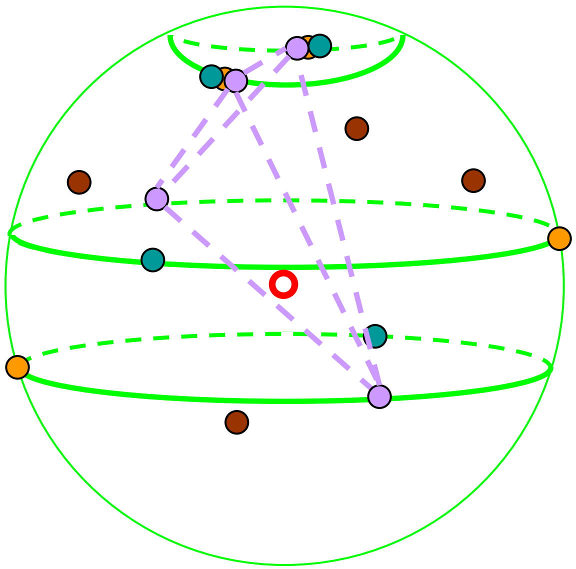

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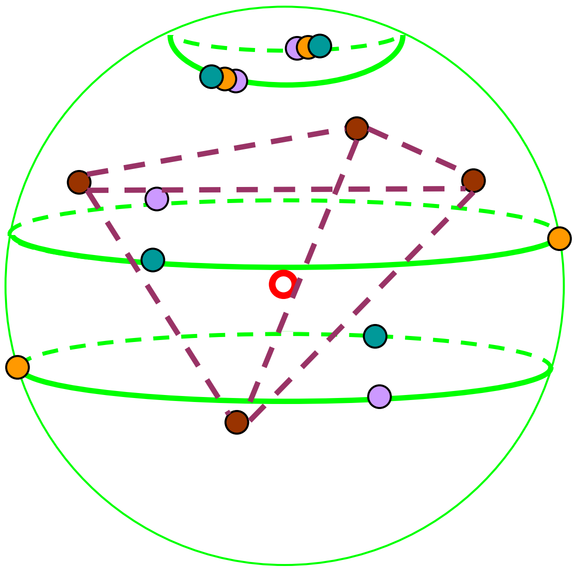

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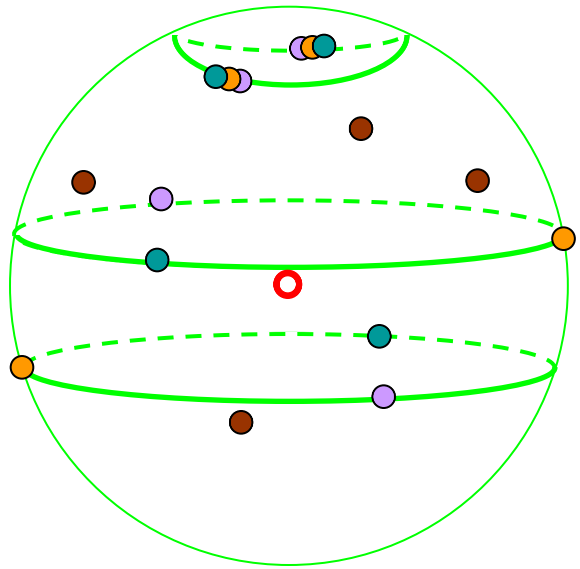
# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

$depth_S(p) = 10$



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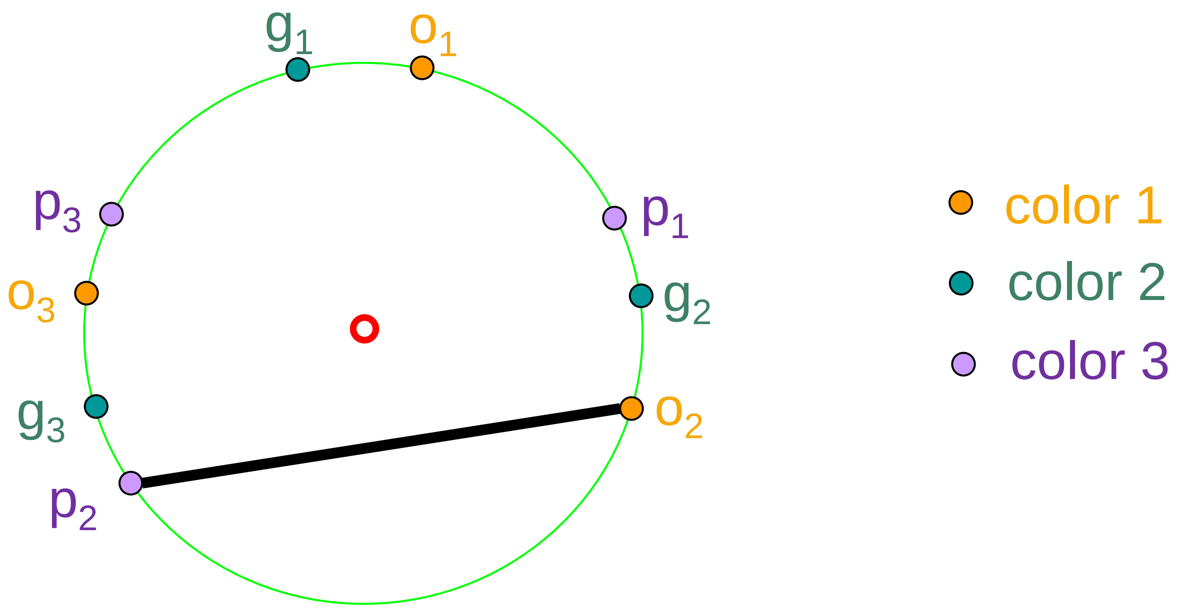
# *Transversal*

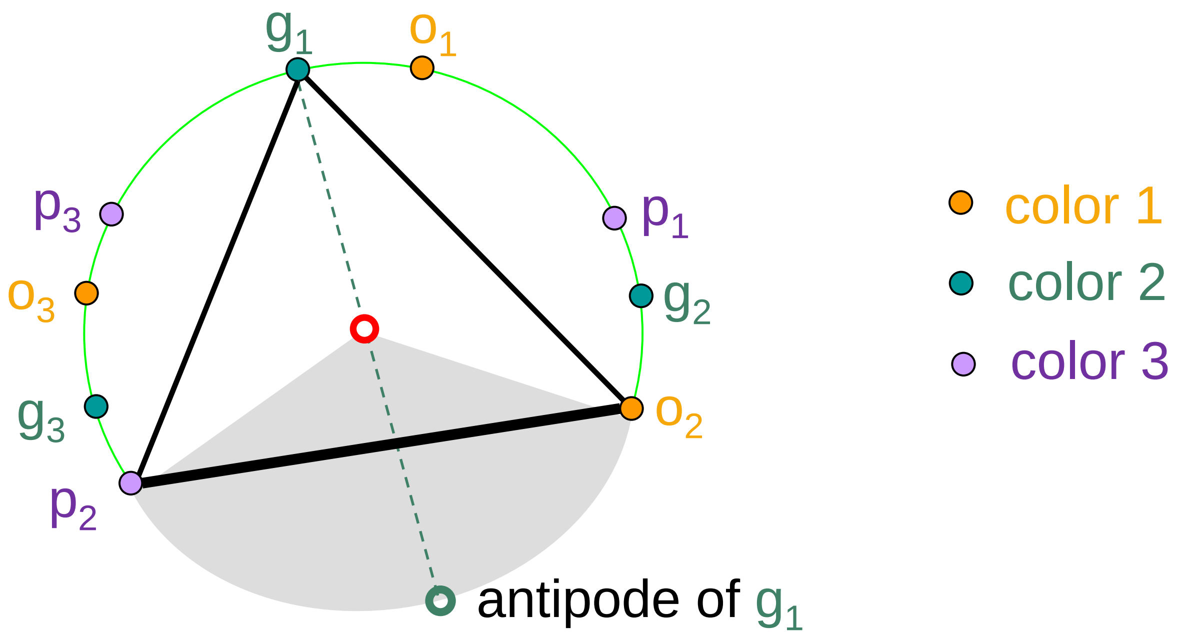
colourful set of $d$ points (one colour missing)




$\hat{2}$-transversal ($o_2$,$p_2$)

# *Transversal*

colourful set of $d$ points (one colour missing)

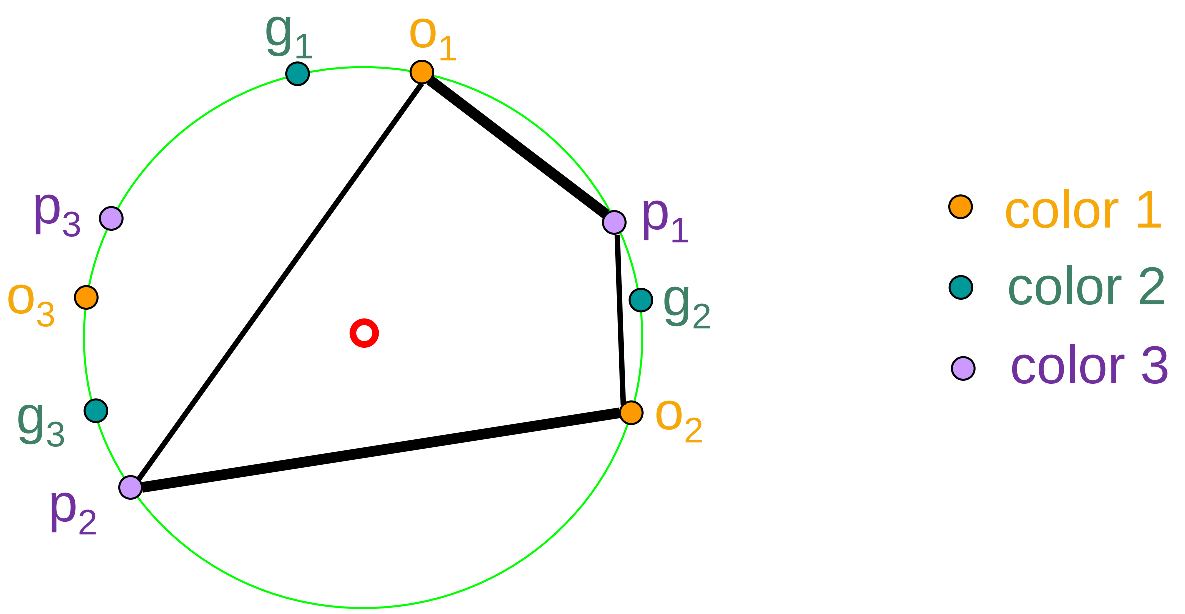



$\hat{2}$-transversal ($o_2$,$p_2$) *spans* the antipode of $g_1$

iff ($o_2$,$p_2$,$g_1$) is a colourful simplex

# *Combinatorial (topological) Octahedra*

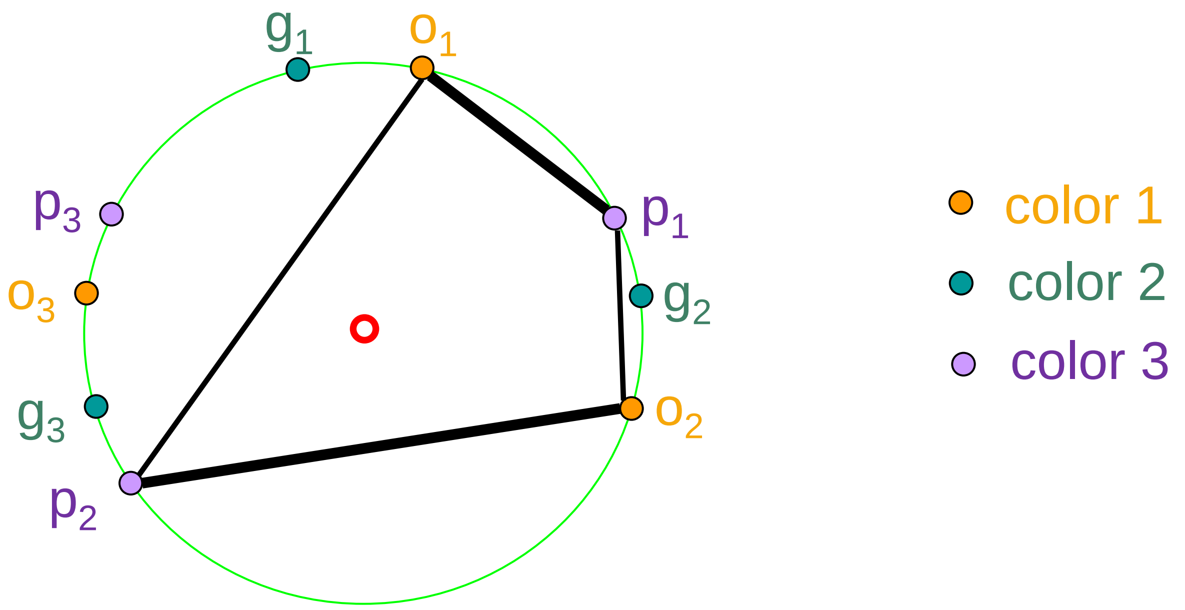
pair of disjoint $\hat{i}$-transversals




octahedron [($o_1$,$p_1$),($o_2$,$p_2$)]

# *Octahedron Lemma*

origin-*containing* octahedra

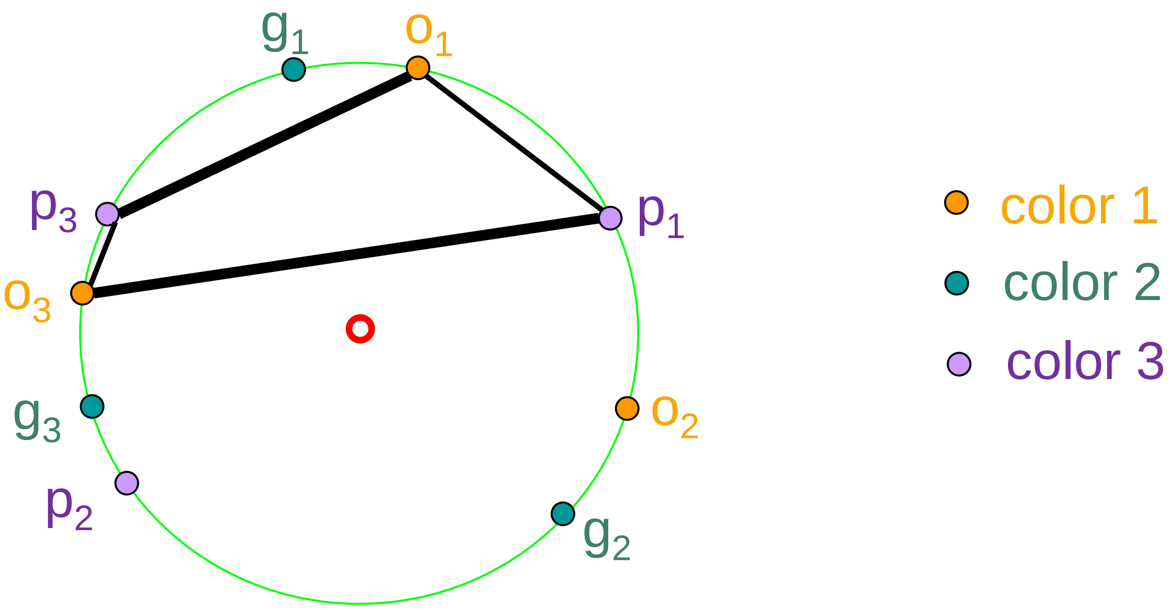



octahedron [($o_1$,$p_1$),($o_2$,$p_2$)]

$2^d$ colourful faces span the whole sphere if it contains the origin (creating $d$+1 colourful simplexes)

# *Octahedron Lemma*

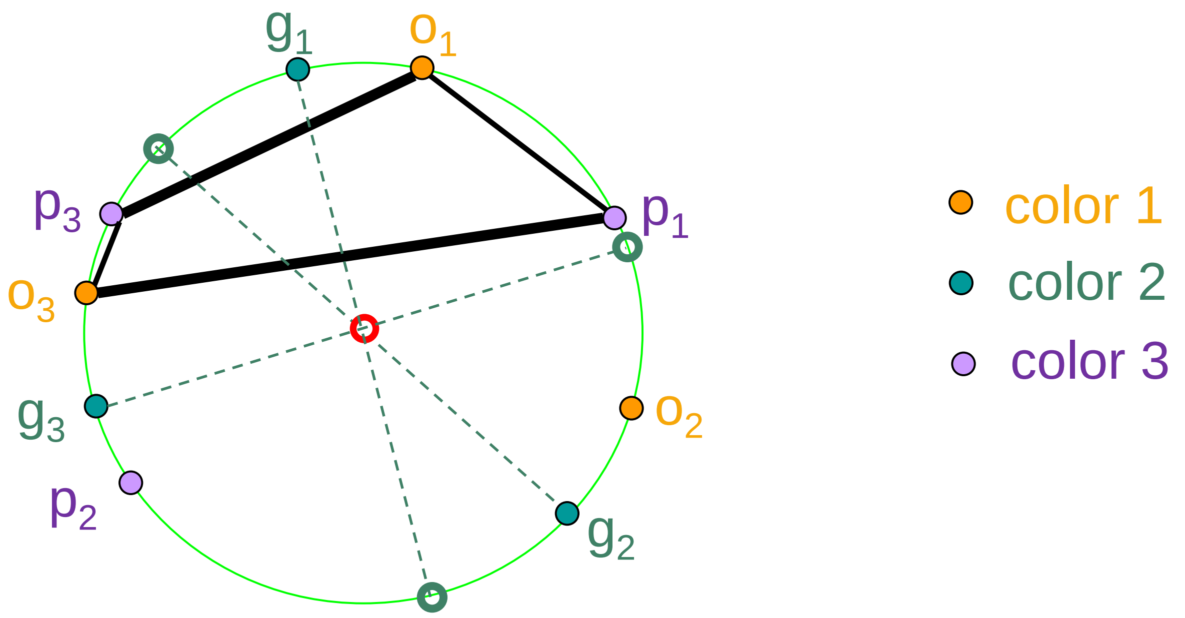
octahedron *not containing* the origin




octahedron [($o_1$,$p_3$),($o_3$,$p_1$)] does not contain $p$

# *Octahedron Lemma*

octahedron *not containing* the origin

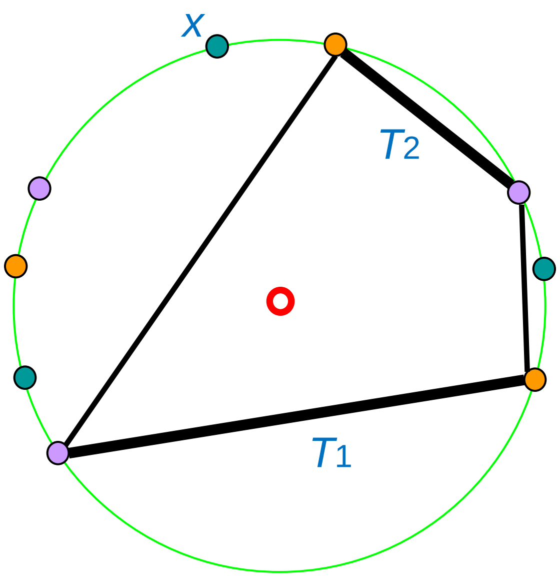


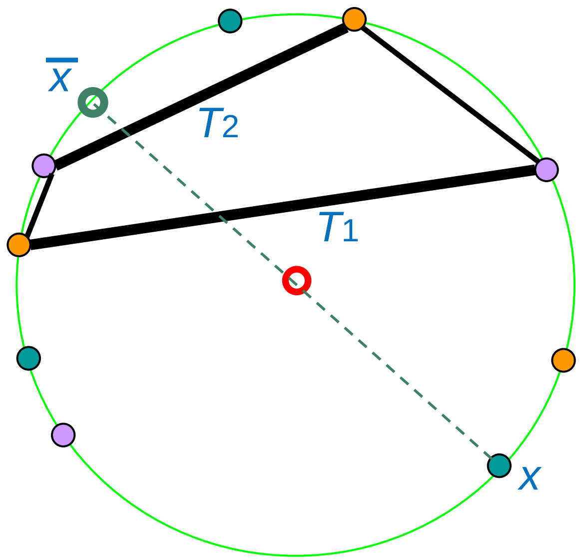

octahedron [($o_1$,$p_3$),($o_3$,$p_1$)] spans any antipode an *even* number of times

# *Octahedron Lemma*

Given 2 disjoint transversals $T_1$ and $T_2$, and $T_1$ spans $\bar{x}$ *(antipode of $x$),*

- ➢ either octahedron ($T_1$,$T_2$) contains $p$,
- ➢ or there exists a transversal $T \neq T_1$ consisting of points from $T_1$ and $T_2$ that spans $\bar{x}$.



# Colourful Research Directions

- Generalize the sufficient condition of Bárány for the existence of a colourful simplex
- Improve lower bound for $\mu(d) = \min_{S,p} depth_S(p)$
- Computational approaches for $\mu(d)$ for small *d*.
- Obtain an efficient algorithm to find a colourful simplex : *Colourful Linear Programming Feasibility* problem

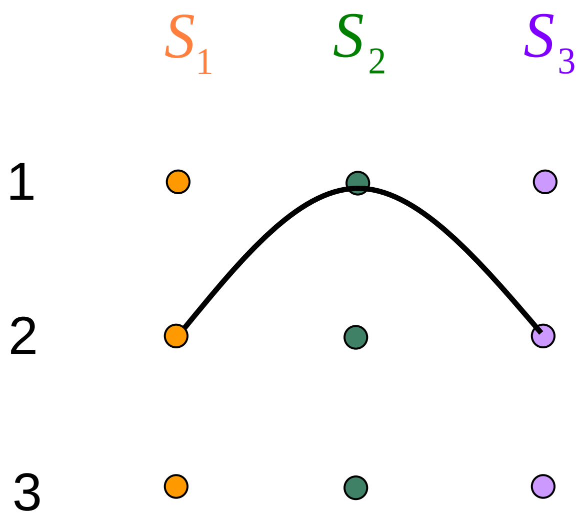
$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

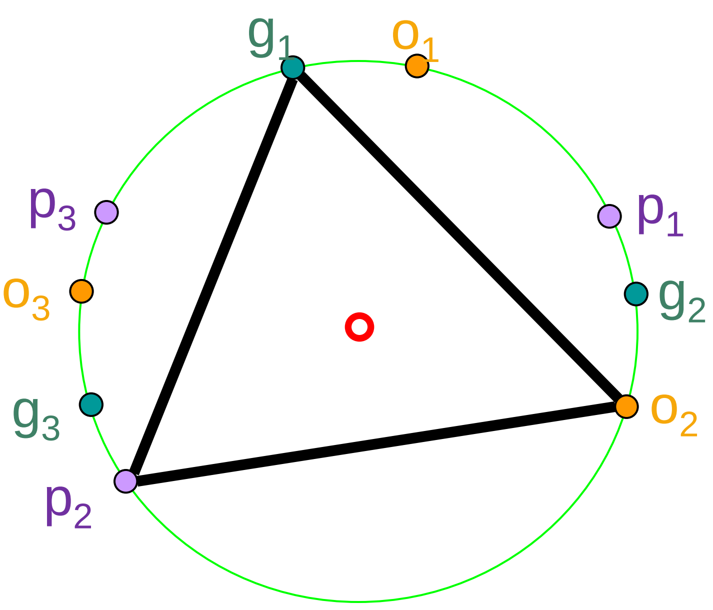
$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \ldots \cap \text{conv}(S_{d+1})$

***S***, *p general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \geq d+1$

# Computational Approach

($d$+1)-uniform ($d$+1)-partite *hypergraph* representation
of *colourful point configurations*




edge: colourful simplex containing *p*

❖ *combinatorial setting suggested by Imre Bárány*

# *Computational Approach*

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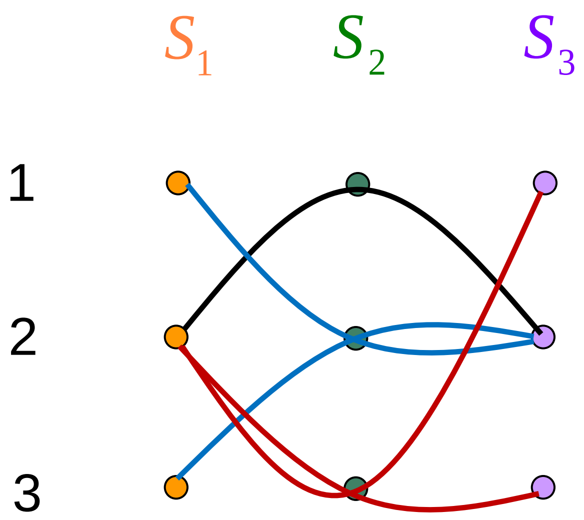
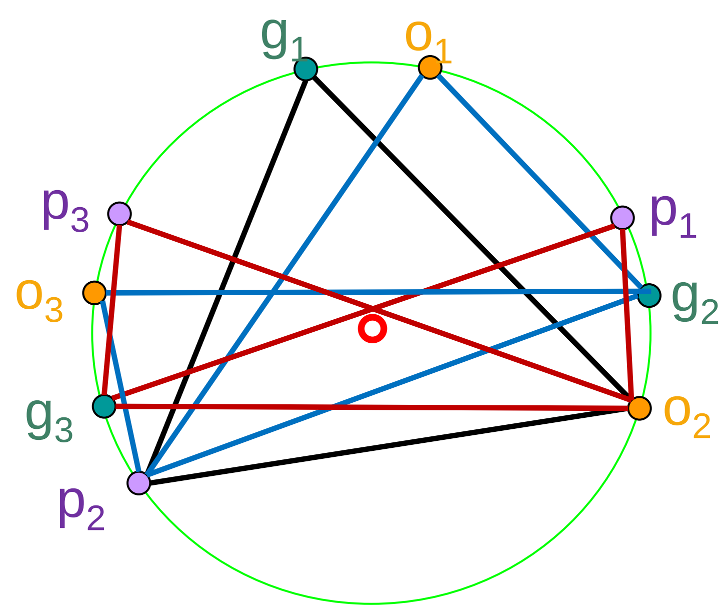


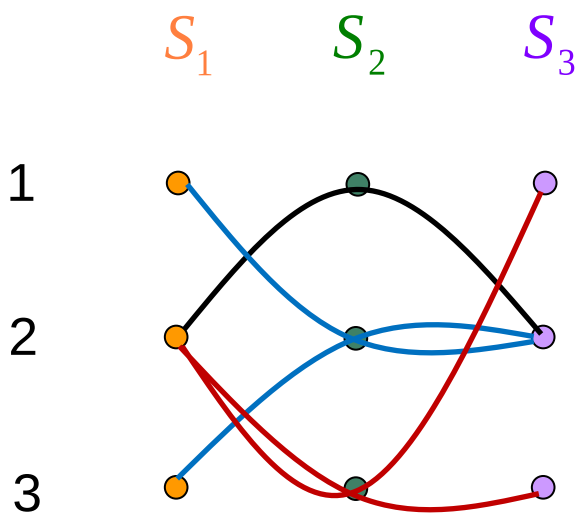

*necessary conditions*:

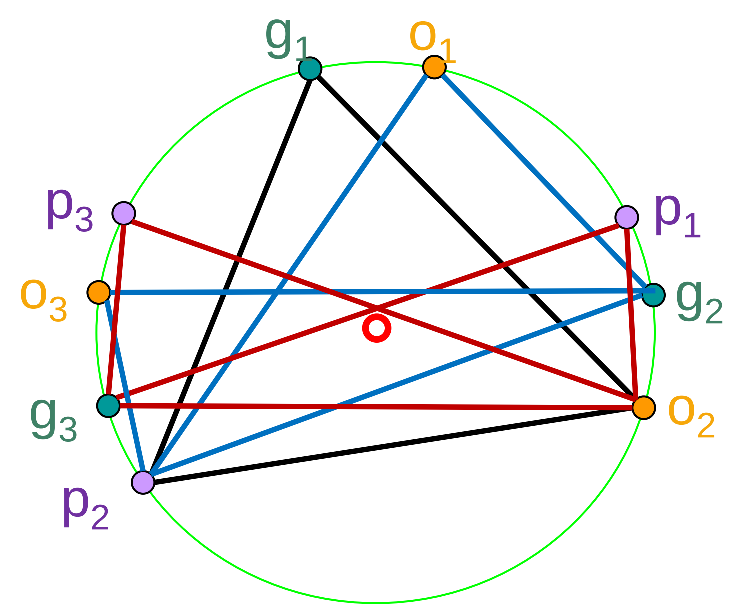
- every vertex belongs to at least 1 edge.
- even number of edges induced by subsets $X_i$ of $S_i$ of size 2

❖ *reformulation of the Octahedron Lemma*

# *Computational Approach*

(*d*+1)-uniform (*d*+1)-partite *hypergraph* representation of *colourful point configurations*



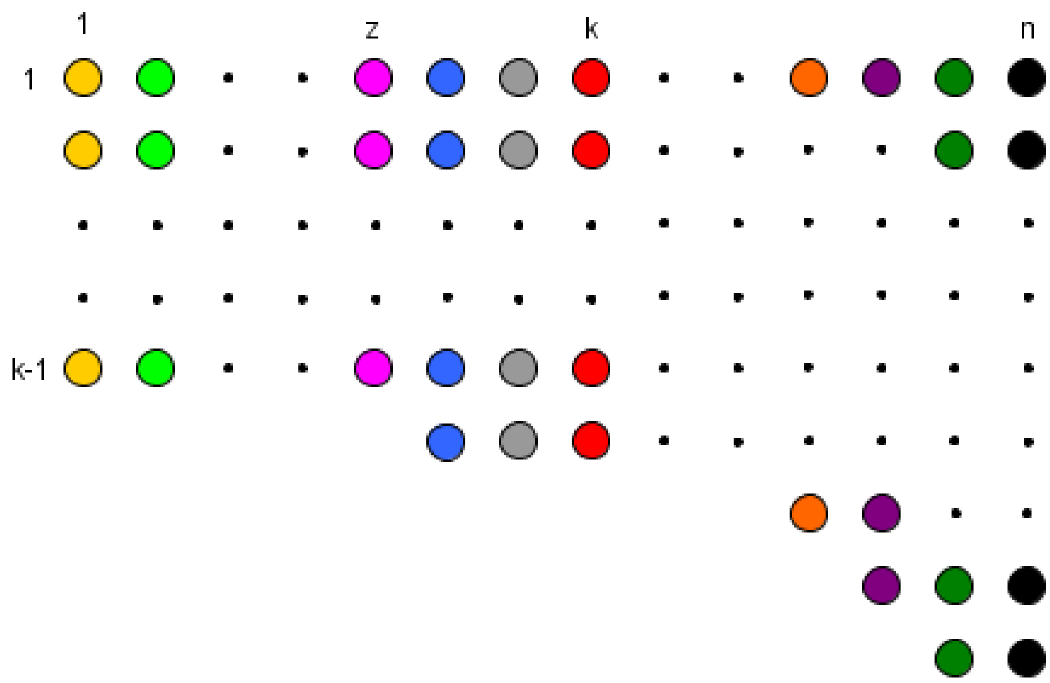

if no hypergraph with $t$ or less hyper-edges satisfies the 2 necessary conditions, then $\mu(d) > t$

$\Rightarrow$ computational proof that $\mu(4) \geq 14$ [D., Stephen, Xie 2013]

❖ *isolated edge argument needed*

# Octahedral Systems

$n$-uniform $n$-partite hypergraph ($S_1$,...,$S_n$ ,E) with $|S_i|\geq 2$ such that the number of edges induced by subsets $X_i$ of $S_i$ of size 2 for $i$=1,...$n$ is even



# *Octahedral Systems*

- *even number of edges* if all $|S_i|$ are even for $i = 1, \ldots, n$
- *symmetric difference* of 2 octahedral systems is octahedral
- existence of *non-realizable* octahedral system
  without isolated vertex
- *number* of octahedral systems: $2^{\prod_1^n |S_i| - \prod_1^n (|S_i|-1)}$

[D., Meunier, Sarrabezolles 2012]

# *Octahedral Systems*

- octahedral system without isolated vertex, $|S_1| = ... = |S_n| = m$ has at least $m(m+5)/2 - 11$ edges, implying:

$$\mu(d) \geq (d+1)(d+6)/2 - 11$$

- further analysis: $\mu(4) = 17$

[D., Meunier, Sarrabezolles 2012]

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[Bárány, Onn 1997] and [D., Huang, Stephen, Terlaky 2008]

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$$(d+1)(d+6)/2 - 11 \leq \mu(d) \leq d^2 + 1 \quad \text{for } d \geq 5$$

$$\mu(d) \text{ even for odd } d$$

$$22 \leq \mu(5) \leq 26$$

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✓ ***thank you***

# Tverberg Theorem

$n$ points can be partitioned into $\left\lfloor \frac{n-1}{d+1} \right\rfloor + 1$ colours, with a point $p$ in convex hull intersection. [Tverberg 1966]

$\binom{\left\lfloor \frac{n-1}{d+1} \right\rfloor + 1}{d+1}$ combinations to choose $d$+1 *colours*.

If each combination has at least $\mu$ colourful simplices. [Bárány 82]

$$\max_p depth_S(p) \geq \mu \binom{\left\lfloor \frac{n-1}{d+1} \right\rfloor + 1}{d+1} = \mu \binom{n}{d+1} + O(n^d)$$

$S$, $p$ *general position*