Erdős-Ko-Rado theorems in geometrical settings

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CanaDAM – June 10-13, 2013

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Overview

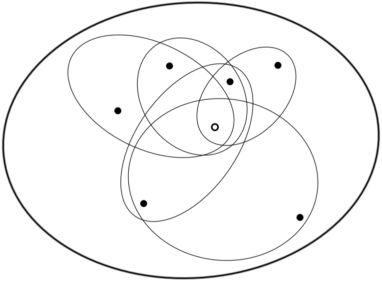
The original Erdős-Ko-Rado problem

Theorem (Erdős-Ko-Rado)

Let Ω be a set of size n and S a family of subsets of size k such that the elements of S are pairwise not disjoint. If $n \geq 2k$, then $|S| \leq \binom{n-1}{k-1}$.

Theorem (Erdős-Ko-Rado)

Let Ω be a set of size n and S a family of subsets of size k such that the elements of S pairwise meet in t elements, $1 \leq t \leq k \leq n$. If $n \geq t + (k-t)\binom{k}{t}^3$, then $|S| \leq \binom{n-t}{k-t}$.



Erdős-Ko-Rado sets and t-intersecting sets

Definition

Let Ω be a finite set of size n. If $\mathcal{S}$ is a family of subsets of fixed size k, such that the elements of $\mathcal{S}$ are pairwise not disjoint, then $\mathcal{S}$ is an Erdős-Ko-Rado set of k-subsets of Ω. If it is non-extendable regarding these conditions, it is called maximal.

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Definition

Let Ω be a finite set of size n. If $\mathcal{S}$ is a family of subsets of fixed size k, such that the elements of $\mathcal{S}$ meet pairwise in at least t elements, $1 \leq t \leq k \leq n$, then $\mathcal{S}$ is a t-intersecting set of k-subsets of Ω.

Erdős-Ko-Rado sets for projective spaces

Notation

- PG(n, q): the n-dimensional projective geometry over $\mathbb{F}_q$.
- $\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{q^{n+1-i}-1}{q^i-1}$: the number of $(k-1)$-spaces in PG$(n-1, q)$.

Definition

If $\mathcal{S}$ is a set of k-spaces in PG(n, q) which are pairwise intersecting non-trivially, then it is an Erdős-Ko-Rado set of k-spaces, briefly an EKR(k) set, in PG(n, q). If $\mathcal{S}$ cannot be extended to a larger set of k-spaces with this property, then $\mathcal{S}$ is called maximal.

Finite classical polar spaces

Definition

Finite classical polar spaces are incidence structures, consisting of the subspaces of a vector space $V(n, q)$ that are totally isotropic with respect to a certain non-degenerate sesquilinear or quadratic form f. The incidence relation is defined by the inclusion relation.

We consider the embedding of the polar spaces in the projective space $\mathrm{PG}(n, q)$. Dimensions are therefore projective dimensions.

Definition

The maximal subspaces of a polar space are called generators (or maximals). If the generators of a polar space have dimension $d-1$, then the polar space has rank d. If a finite polar space is embedded in $\mathrm{PG}(n, q)$ and there are q^e+1 different generators through a $(d-2)$-space, then this polar space has parameter e.

Erdős-Ko-Rado sets for polar spaces

List of finite classical polar spaces of rank d:

- the quadrics $\mathcal{Q}^+(2d-1,q)$ $(e=0)$, $\mathcal{Q}(2d,q)$ $(e=1)$ and $\mathcal{Q}^-(2d+1,q)$ $(e=2)$;
- the Hermitian varieties $\mathcal{H}(2d-1,q)$ $(e=1/2)$ and $\mathcal{H}(2d,q)$ $(e=3/2)$, q a square;
- the symplectic polar spaces $\mathcal{W}(2d-1,q)$ $(e=1)$.

From now on, a polar space is always classical and finite.

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From now on, a polar space is always classical and finite.

Definition

Let $\mathcal{P}$ be a polar space of rank d. If $\mathcal{S}$ is a set of k-spaces on $\mathcal{P}$ which are pairwise non-disjoint, $1 \leq k < d$, then it is called an Erdős-Ko-Rado set of k-spaces, briefly an EKR(k) set, on $\mathcal{P}$. If $\mathcal{S}$ cannot be extended to a larger set of k-spaces with this property, then $\mathcal{S}$ is called maximal.

Erdős-Ko-Rado sets for block designs

Definition

A $t-(v,k,\lambda)$ design $\mathcal{D}=(P,B,I)$ is an incidence structure consisting of a set P of v points and a set B of blocks consisting of k points, such that precisely λ blocks are incident with t given points.

Definition

Let $\mathcal{D}$ be a $t-(v,k,\lambda)$ design. If $\mathcal{S}$ is a set of blocks of $\mathcal{D}$, pairwise meeting in s points, $1 \leq s < t$, then it is called an s-intersecting set of $\mathcal{D}$. If $s=1$, it is called an Erdős-Ko-Rado set.

Overview

1 Introduction

2 Results about the largest EKR sets

3 Classification theorems for maximal EKR sets

Erdős-Ko-Rado theorems for sets

Theorem (Wilson)

Let $1 \leq t \leq k$ be positive integers and let $\mathcal{S}$ be a t-intersecting set of subsets of size k in a set Ω with $|\Omega| = n$. If $n \geq (t+1)(k-t+1)$, then $|\mathcal{S}| \leq \binom{n-t}{k-t}$. Moreover, if $n \geq (t+1)(k-t+1)+1$, then equality holds if and only if $\mathcal{S}$ is the set of all subsets of size k through a fixed t-subset of Ω.

Corollary

Let k be a positive integer and let $\mathcal{S}$ be an Erdős-Ko-Rado set of subsets of size k in a set Ω with $|\Omega| = n$. If $n \geq 2k$, then $|\mathcal{S}| \leq \binom{n-1}{k-1}$. Moreover, if $n \geq 2k+1$, then equality holds if and only if $\mathcal{S}$ is the set of all subsets of size k through a fixed element of Ω.

Erdős-Ko-Rado theorems for PG(n, q)

Theorem (Hsieh, Frankl-Wilson, ..., Tanaka)

If $\mathcal{S}$ is an EKR(k) set in PG(n, q)*, with $n \geq 2k+1$, then $|\mathcal{S}| \leq \begin{bmatrix} n \\ k \end{bmatrix}_q$. In case of equality, $\mathcal{S}$ is the set of k-spaces through a fixed point, or $n = 2k+1$ and $\mathcal{S}$ consists of all k-spaces in a fixed hyperplane.*

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Theorem (Tanaka)

Let $0 \leq t \leq k$ be positive integers. If $\mathcal{S}$ is a set of k-dimensional subspaces in PG(n, q)*, with $n \geq 2k+1$, pairwise intersecting in at least a t-dimensional subspace, then $|\mathcal{S}| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$. Equality holds if and only if $\mathcal{S}$ is the set of all subspaces with dimension k, containing a fixed t-dimensional subspace of* PG(n, q)*, or $n = 2k+1$ and $\mathcal{S}$ is the set of all subspaces with dimension k in a fixed $(n-t-1)$-dimensional subspace.*

Erdős-Ko-Rado theorems for generators in polar spaces

Theorem (Pepe-Vanhove-Storme)

The EKR($d-1$) sets, i.e. Erdős-Ko-Rado sets of generators, of maximal size have been classified for all polar spaces of rank d except for the Hermitian varieties $\mathcal{H}(2d-1, q)$ with d odd, q a square.

Results Pepe-Vanhove-Storme: the quadrics

Polar space	Maximum size	Classification
$\mathcal{Q}^-(2n+1,q)$	$(q^2+1)\cdots(q^n+1)$	p.-p.
$\mathcal{Q}(4n,q)$	$(q+1)\cdots(q^{2n-1}+1)$	p.-p.
$\mathcal{Q}(4n+2,q)$, $n \geq 2$	$(q+1)\cdots(q^{2n}+1)$	p.-p., one class $\mathcal{Q}^+(4n+1,q)$
$\mathcal{Q}(6,q)$	$(q+1)(q^2+1)$	p.-p., one class $\mathcal{Q}^+(5,q)$, base plane
$\mathcal{Q}^+(4n+1,q)$	$(q+1)\cdots(q^{2n}+1)$	one class
one class $\mathcal{Q}^+(4n+3,q)$, $n \geq 2$	$(q+1)\cdots(q^{2n}+1)$	p.-p.
one class $\mathcal{Q}^+(7,q)$	$(q+1)(q^2+1)$	p.-p., meeting Greek in a plane

p.-p.: point-pencil, all generators through a fixed point.

Results Pepe-Vanhove-Storme: Hermitian and symplectic

Polar space	Maximum size	Classification
$\mathcal{W}(4n+1,q)$, $n \geq 2$, q odd	$(q+1)\cdots(q^{2n}+1)$	p.-p.
$\mathcal{W}(5,q)$, q odd	$(q+1)(q^2+1)$	p.-p., base plane
$\mathcal{W}(4n+1,q)$, $n \geq 2$, q even	$(q+1)\cdots(q^{2n}+1)$	p.-p., Latins $\mathcal{Q}^+(4n+1,q)$
$\mathcal{W}(5,q)$, q even	$(q+1)(q^2+1)$	p.-p., base plane Latins $\mathcal{Q}^+(5,q)$
$\mathcal{W}(4n+3,q)$	$(q+1)\cdots(q^{2n+1}+1)$	p.-p.
$\mathcal{H}(2n,q^2)$	$(q^3+1)(q^5+1)\cdots(q^{2n-1}+1)$	p.-p.
$\mathcal{H}(4n+3,q^2)$	$(q+1)(q^3+1)\cdots(q^{4n+1}+1)$	p.-p.
$\mathcal{H}(4n+1,q^2)$, $n \geq 2$	$< \frac{(q+1)(q^3+1)\cdots(q^{4n+1}+1)}{q^{2n+1}+1}$ (*)	?
$\mathcal{H}(5,q^2)$	$q(q^4+q^2+1)+1$	base plane

(*) This bound is $\Theta(q^{4n^2+2n})$, but was improved by Ihringer and Metsch to $\Theta(q^{4n^2+1})$.

Erdős-Ko-Rado theorems for block designs

Theorem (Rands)

Let $\mathcal{D}$ be a $t-(v,k,\lambda)$ block design and let $\mathcal{S}$ be an s-intersecting set of $\mathcal{D}$, $0 < s < t \leq k$.

- *If $s < t-1$ and $v \geq s + \binom{k}{s}(k-s+1)(k-s)$, or*
- *if $s = t-1$ and $v \geq s + \binom{k}{s}^2(k-s)$,*

then $|\mathcal{S}| \leq \lambda_s$ and equality is obtained if and only if $\mathcal{S}$ is the set of blocks through s fixed points.

Theorem (De Boeck)

Let $\mathcal{D}$ be a $2-(v,k,1)$ design, $k \geq 4$, with $r = \frac{v-1}{k-1} \geq k^2 - 3k + \frac{3}{4}\sqrt{k} + 2$, and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$. Then $|\mathcal{S}| \leq r$. If $r \neq k^2 - k + 1$ and $(r,k) \neq (8,4)$, then equality is obtained if and only if $\mathcal{S}$ is a point-pencil.

Overview

The general Erdős-Ko-Rado problem

Aim: find all maximal Erdős-Ko-Rado sets in a geometry $\mathcal{G}$, with size at least s.

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Some easy classification theorems of maximal EKR sets:

- There are precisely two types of maximal Erdős-Ko-Rado sets of subsets of size 2 in a finite set.
- There are precisely two types of maximal EKR(1) sets in PG(n, q).
- Let $\mathcal{P}$ be a polar space of rank $d \geq 2$. If $d = 2$, then there is only one type of maximal EKR(1) sets. If $d \geq 3$, then there are precisely two types of maximal EKR(1) sets.
- If $n \leq 2k$, there is only one type of EKR(k) sets in PG(n, q).

Hilton-Milner

Theorem (Hilton-Milner)

Let Ω be a set of size n and let $\mathcal{S}$ be an Erdős-Ko-Rado set of subsets of size k, $k \geq 3$ and $n \geq 2k+1$. If there is no element in Ω which is contained in all subsets of $\mathcal{S}$, then

$$|\mathcal{S}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1 .$$

Moreover, equality holds if and only if

- *either $\mathcal{S}$ is the union of a fixed k-subset F and the set of all k-subsets G of Ω containing a fixed element $x \notin F$, such that $G \cap F \neq \emptyset$,*
- *or else $k = 3$ and $\mathcal{S}$ is the set of all subsets of size 3 having an intersection of size at least 2 with a fixed subset F of size 3.*

Hilton-Milner in PG(n, q)

Theorem (Blokhuis-Brouwer-Chowdury-Frankl-Mussche-Patkos-Szőnyi)

Let $\mathcal{S}$ be an EKR(k) set in PG(n, q), with $k \geq 2$, $n \geq 2k+2$ and $q \geq 3$ (or $n \geq 2k+4$, $k \geq 2$ and $q = 2$). If the k-spaces of $\mathcal{S}$ do not go through a fixed common point, then

$$|\mathcal{S}| \leq \begin{bmatrix} n \\ k \end{bmatrix}_q - q^{k(k+1)} \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q + q^{k+1}.$$

Moreover, if equality holds, then

- *either $\mathcal{S}$ consists of all k-spaces through a fixed point P, meeting a fixed $(k+1)$-space τ in a j-dimensional subspace, $j \geq 1$, and all k-spaces in τ,*
- *or else $k = 2$ and $\mathcal{S}$ is the set of all planes (2-spaces) meeting a fixed plane π in at least a line (1-space).*

Classification results for EKR sets in unitals

Theorem (De Boeck)

Let $\mathcal{U}$ be a unital without the O'Nan configuration (e.g. classical unitals) and let $\mathcal{S}$ be a maximal Erdős-Ko-Rado set on $\mathcal{U}$. Then, $\mathcal{S}$ is a point-pencil or a triangle (set of blocks through a fixed point meeting a fixed block, together with this fixed block).

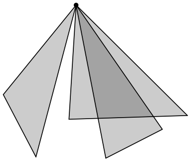
Theorem (De Boeck)

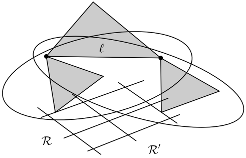
Let $\mathcal{U}$ be a non-classical unital of order q and let $\mathcal{S}$ be a maximal Erdős-Ko-Rado set on $\mathcal{U}$. If $q \geq 5$, then either $|\mathcal{S}| = q^2$ and $\mathcal{S}$ is a point-pencil, or else $|\mathcal{S}| \leq q^2 - q + \sqrt[3]{q^2} - \frac{2}{3}\sqrt[3]{q} + 1$. If $q = 4$, then either $|\mathcal{S}| = 16$ and $\mathcal{S}$ is a point-pencil, or else $|\mathcal{S}| \leq 13$. If $q = 3$, then either $|\mathcal{S}| = 9$ and $\mathcal{S}$ is a point-pencil, or else $|\mathcal{S}| \leq 8$.

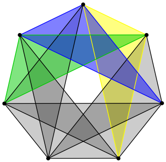
A classification theorem for EKR(2) sets

Theorem (De Boeck)

Let $\mathcal{P}$ be a projective space of dimension at least 5 or classical polar space of rank at least 3 and let $\mathrm{PG}(n,q)$ be the ambient space of $\mathcal{P}$. Let $\mathcal{S}$ be a maximal EKR(2) set in $\mathcal{P}$. Then $\mathcal{S}$ belongs to one of eleven described types or is contained in a 5-space of $\mathrm{PG}(n,q)$.







EKR(2) sets in PG(n, q) and polar spaces of large rank

Theorem (Blokhuis-Brouwer-Szőnyi, De Boeck)

Let $\mathcal{S}$ be a maximal EKR(2) set in PG(n, q), $n \geq 5$, *with* $|\mathcal{S}| \geq 3q^4 + 3q^3 + 2q^2 + q + 1$.

- *If $n = 5, 6$, then $\mathcal{S}$ belongs to one of six described types.*
- *If $n \geq 7$, then $\mathcal{S}$ belongs to one of ten described types.*

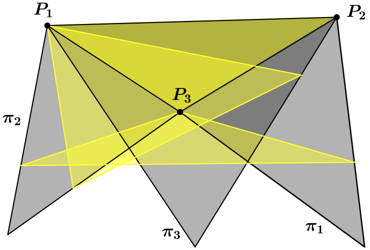
Theorem (De Boeck)

Let $\mathcal{P}$ be a polar space of rank d and let $\mathcal{S}$ be a maximal EKR(2) set on $\mathcal{P}$ with $|\mathcal{S}| \geq 3q^4 + 3q^3 + 2q^2 + q + 1$. If $d \geq 6$, then $\mathcal{S}$ belongs to one of twelve described types.

EKR(2) sets in polar spaces of small rank

Theorem (Brouwer-Hemmeter, De Boeck)

For $\mathcal{Q}^+(2d-1,q)$, $\mathcal{Q}(2d,q)$, $\mathcal{Q}^-(2d+1,q)$ and $\mathcal{W}(2d-1,q)$, $3 \leq d \leq 5$, all EKR(2) sets have been classified.



Theorem (Pepe-Storme-Vanhove, De Boeck)

For $\mathcal{H}(2d-1,q)$ and $\mathcal{H}(2d,q)$, q a square and $3 \leq d \leq 5$, all EKR(2) sets with size at least $q^2\sqrt{q}+q\sqrt{q}+\sqrt{q}+1$ have been classified.

Open problems

- Largest EKR(k) on polar space of rank d, with $2 < k < d - 1$?
- Improving results Blokhuis-Brouwer-Szőnyi on EKR(2) in PG$(5, q)$.
- Largest t-intersecting set of generators on polar spaces?
- ...

Thank you for your attention.