Unrolling residues to avoid progressions

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Coloring 1, . . . , 28

Take the numbers 1 through 28 and place them in two groups (i.e., color with two colors Red and Blue). And consider the number of monochromatic triples of equally spaced terms.

- How to maximize? Easy. We color everything red. 182 such triples.
- How many should we expect at random? Easy. Each triple has probability of 1/4 of being monochromatic so at random expect 45.5.
- How to minimize? Hard. But thankfully 28 is small!

RRRRBBBRRRRBBBRRBRRRRRRRRRRBBBRRRRBBB
Coloring 1, \ldots, n

Given that

- we are coloring 1, \ldots, n
- using \( r \) different colors
- trying to avoid monochromatic \( k \)-APs

then what is the best we can do?

How few can we get?
What kind of pattern achieves this?

**k-APs**

A **k-term arithmetic progression** are \( k \) equally spaced integers, i.e., \( a, a + d, \ldots, a + (k - 1)d \).
Theorem (van der Waerden)
For any number \( r \) of colors and \( k \) of length of arithmetic progressions there is a threshold \( N \) so that if \( n \geq N \) then any coloring of \( 1, 2, \ldots, n \) using \( r \) colors must have a monochromatic arithmetic progression of length \( k \).

Theorem (Frankl-Graham-Rödl)
For fixed \( r \) and \( k \), there is \( c > 0 \) so that the number of monochromatic \( k \)-APs in any \( r \)-coloring of \( 1, 2, \ldots, n \) is at least \( cn^2 + o(n^2) \).
How about random?

Observation
The number of $k$-APs in $1, 2, \ldots, n$ is
\[
\frac{(n - a)(n - k + 1 + a)}{2(k - 1)} = \frac{n^2}{2(k - 1)} + O(n),
\]
where $n = (k - 1)\ell + a$ and $0 \leq a < k - 1$.

Lemma
There is a coloring of $1, 2, \ldots, n$ with $r$ colors which has at most $\frac{1}{2(k-1)r^{k-1}}n^2 + O(n)$ monochromatic $k$-APs.
Proof: Color randomly. \qed
Expanding a coloring

Start with a good small coloring and expand it, i.e.,

\[ \text{RRRRRRBBBBRRRRRRBBBBBBBBBBBBRRRRRRRRRRRRBBBBBBRRRBBBBBB} \]

For \( n \) large this gives \( \frac{3}{56} n^2 + O(n) \) monochromatic 3-APs, or 21.42\% (random is 25\%).

Theorem (Parrilo-Robertson-Saracino; Butler-Costello-Graham)
Expanding the following coloring gives \( \frac{117}{2192} n^2 + O(n) \) monochromatic 3-APs, or 21.35\%:

\[ \text{R} \cdot \text{RB} \cdot \text{BR} \cdot \text{RB} \cdot \text{BR} \cdot \text{RB} \cdot \text{BR} \cdot \text{RB} \cdot \text{BR} \cdot \text{RB} \cdot \text{BR} \cdot \text{RB} \cdot \text{BR} \cdot \text{RB} \cdot \text{BR} \cdot \text{RB} \cdot \text{RB} \cdot \text{B} \]

\[ 28 \ 6 \ 28 \ 37 \ 59 \ 116 \ 116 \ 59 \ 37 \ 28 \ 6 \ 28 \]
Unrolling a coloring

Start with a good small coloring and repeat, i.e.,

\[ \text{RRRBBBRRBBB} \text{BBBBBBRRRRRBBBBRRBBBRRBBRBBBRRRBBBBRRBBBRRBB} \]

For \( n \) large this gives \( \frac{1}{16} n^2 + O(n) \) monochromatic 3-APs, the same as random!

Repeating \emph{any} pattern will \emph{not} beat the random bound for 3-APs... but for \( k \)-APs with \( k \geq 4 \) the story is very different!
Good coloring of \( \mathbb{Z}_{11} \)

Lu and Peng found the following good coloring of \( \mathbb{Z}_{11} \):

\[
\begin{array}{cccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & n = 11 \\
\end{array}
\]

- 4-AP free

Then unrolled it to get a coloring of the integers:
Good coloring of $\mathbb{Z}_{11}$

$$\ell = \sum_i b_i \cdot 11^i \quad \text{where} \quad 0 \leq b_i \leq 10.$$  

Let $j$ be the smallest index so that $b_j \neq 0$,

\[
\text{color } \ell \quad \left\{ \begin{array}{ll}
\text{red} & \text{if } b_j = 1, 3, 4, 5, \text{ or } 9; \\
\text{blue} & \text{if } b_j = 2, 6, 7, 8, \text{ or } 10.
\end{array} \right.
\]

This coloring has $\frac{1}{72} n^2 + O(n)$ monochromatic 4-APs. (Far superior to the best coloring found by expanding blocks.)
Theorem
If there is a coloring of $\mathbb{Z}_m$ with $r$ colors which have no monochromatic $k$-APs and 0 can be colored arbitrarily, then there is an $r$-coloring of $1, 2, \ldots, n$ which has
$$\frac{1}{2(m+1)(k-1)}n^2 + O(n)$$ monochromatic $k$-APs.

Proof: After unrolling we have $m - 1$ monochromatic arithmetic progressions of length $n/m$. The last residue we recursively color.
Let $F(n)$ be the number of monochromatic $k$-APs then (ignoring lower order terms)
$$F(n) = F\left(\frac{n}{m}\right) + \frac{m-1}{2(k-1)}\left(\frac{n}{m}\right)^2 = \frac{(m-1)n^2}{2(k-1)} \sum_{i \geq 1} \left(\frac{1}{m^2}\right)^i$$
$$= \frac{(m-1)n^2}{2(k-1)} \cdot \frac{1}{m^2 - 1} = \frac{1}{2(m+1)(k-1)} n^2.$$
\qed
What do we unroll?

color $\ell \begin{cases} \text{red} & \text{if } b_j = 1, 3, 4, 5, \text{ or } 9; \\ \text{blue} & \text{if } b_j = 2, 6, 7, 8, \text{ or } 10. \end{cases}$

Note that $\{1, 3, 4, 5, 9\}$ are the quadratic residues of $\mathbb{Z}_{11}$!

Quadratic residues are good for 2 colors:

- Easy to see if $k$-AP free (i.e., only have to check for longest run of residues).
- Longest run cannot be too big: $O(p^{1/4}(\log p)^{3/2})$ (Burgess)
More generally

For more colors use higher order residues, i.e.,
\( \{x^r \mid x \in \mathbb{Z}_p, x \neq 0 \} \).

**Theorem**

For \( i = 1, 2 \), let \( C_i \) be a coloring of \( \mathbb{Z}_{m_i} \) using \( r_i \) colors where 0 can be colored arbitrarily and containing no nontrivial \( k \)-APs. Then there exists a coloring \( C \) of \( \mathbb{Z}_{m_1 m_2} \) using \( r_1 r_2 \) colors where 0 can be colored arbitrarily and containing no nontrivial \( k \)-APs.
Large $p$ with small runs
### Best known results

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What about $k = 3, r = 3$?

Cannot use residues since $-1, 0, 1$ are all cubic residues. But we don’t have to use residues to do unrolling.

This gives $\frac{1}{48} n^2 + O(n)$ monochromatic 3-APs, so 8.33% of the colorings will be monochromatic, whereas in a random coloring we would expect 11.11% of the 3-APs to be monochromatic.
Open problems/Conclusion

- We have done constructions and found “upper bounds” for the best colorings. What about “lower bounds”.
- For $k = 3$ and $r = 2$ show $\geq \frac{117}{2192} n^2 + O(n)$.
- Show that we can always beat random.
- Is unrolling almost always best?
- Are residues almost always the best thing to unroll?
- What about avoiding non-APs?
- Thank you.