Two Laplacians for the distance matrix of a graph

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1. Adjacency related matrices

**Adjacency matrix**

- For a graph \( G = (V, E) \) on \( n \) vertices, the **adjacency matrix** \( A = A(G) \) is the \( 0 \times 1 \) \( n \times n \)-matrix indexed by the vertices of \( G \) and defined by \( a_{i,j} = 1 \) if and only if \( i \) \( \in E \)

- The **(adjacency) spectrum** \((\lambda_1, \lambda_2, \ldots, \lambda_n)\) of \( G \), with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \), is the \( A \)'s spectrum

\[
A = \begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}
\]

\( A \)-spectrum: \((3, 1, 0, 0, -2, -2)\)
1. Adjacency related matrices

Laplacian matrix

- The **Laplacian** of $G$ is defined by $L = L(G) = \text{Deg} - A$, where $\text{Deg}$ is the diagonal matrix whose diagonal entries are the degrees in $G$, and $A$ the adjacency matrix of $G$.

- The **Laplacian spectrum** $(\mu_1, \mu_2, \ldots, \mu_n)$ of $G$, with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$, is the $L$’s spectrum.

$$L = \begin{bmatrix}
3 & 0 & -1 & -1 & -1 & 0 \\
0 & 3 & 0 & -1 & -1 & -1 \\
-1 & 0 & 3 & 0 & -1 & -1 \\
-1 & -1 & 0 & 3 & 0 & -1 \\
-1 & -1 & -1 & 0 & 3 & 0 \\
0 & -1 & -1 & -1 & 0 & 3 \\
\end{bmatrix}$$

$L$–spectrum : $(5, 5, 3, 3, 2, 0)$
1. Adjacency related matrices

**Signless Laplacian matrix**

- The **signless Laplacian** of $G$ is defined by $Q = Q(G) = \text{Deg} + A$
- The **Laplacian spectrum** $(q_1, q_2, \ldots, q_n)$ of $G$, with $q_1 \geq q_2 \geq \cdots \geq q_n$, is the $Q$’s spectrum

$$Q = \begin{bmatrix}
3 & 0 & 1 & 1 & 1 & 0 \\
0 & 3 & 0 & 1 & 1 & 1 \\
1 & 0 & 3 & 0 & 1 & 1 \\
1 & 1 & 0 & 3 & 0 & 1 \\
1 & 1 & 1 & 0 & 3 & 0 \\
0 & 1 & 1 & 1 & 0 & 3 \\
\end{bmatrix}$$

$Q$–spectrum : $(6, 4, 3, 3, 1, 1)$
2. Distance matrix

**Definition**

- In a connected graph $G$ the **distance** $d(i,j) = d_G(i,j)$ is the length of a shortest path between $i$ and $j$.
- The **distance matrix** $D = D(G)$ of a connected graph $G$ is the $n \times n$-matrix indexed by the vertices of $G$ and where $D_{i,j} = d(i,j)$.
- The **distance spectrum** or $D$-spectrum is denoted by $(\partial_1, \partial_2, \ldots, \partial_n)$ with $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$.

$$D = \begin{bmatrix}
0 & 2 & 1 & 1 & 1 & 2 \\
2 & 0 & 2 & 1 & 1 & 1 \\
1 & 2 & 0 & 2 & 1 & 1 \\
1 & 1 & 2 & 0 & 2 & 1 \\
1 & 1 & 1 & 2 & 0 & 2 \\
2 & 1 & 1 & 1 & 2 & 0 \\
\end{bmatrix}$$

$D$-spectrum : $(7, 0, 0, -2, -2, -3)$
3. Distance Laplacian

**Definition**

- **The transmission** of a vertex $i$ is the sum of all the distances from $i$ to all other vertices $t_i = \sum_{j \in V} d(i, j)$.

- The **distance Laplacian matrix** of $G$ is defined by $\mathcal{D}^L = \text{Tr} - \mathcal{D}$, where $\text{Tr}$ is the diagonal matrix whose diagonal entries are the transmissions in $G$.

- The **distance Laplacian spectrum** or $\mathcal{D}^L$–spectrum is denoted by $(\partial_1^L, \partial_2^L, \ldots, \partial_n^L)$ with $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_n^L = 0$.

\[
\mathcal{D}^L = \begin{bmatrix}
7 & -2 & -1 & -1 & -1 & -2 \\
-2 & 7 & -2 & -1 & -1 & -1 \\
-1 & -2 & 7 & -2 & -1 & -1 \\
-1 & -1 & -2 & 7 & -2 & -1 \\
-1 & -1 & -1 & -2 & 7 & -2 \\
-2 & -1 & -1 & -1 & -2 & 7 \\
\end{bmatrix}
\]

$\mathcal{D}^L$–spectrum : $(10, 9, 9, 7, 7, 0)$
3. Distance Laplacian

### Examples of distance Laplacian spectra

- The complete graph $K_n : \left( n^{(n-1)}, 0 \right)$ (also the Laplacian spectrum)
- The complement of an edge $K_n - e : \left( n + 2, n^{(n-2)}, 0 \right)$
- The star $S_n : \left( 2n - 1^{(n-2)}, n, 0 \right)$
- The complete bipartite graph $K_{a,b} : \left( 2n - a^{(b-1)}, 2n - b^{(a-1)}, n, 0 \right)$
- The complete split graph $SK_{n,\alpha} : \left( n + \alpha^{\alpha-1}, n^{n-\alpha}, 0 \right)$
### Properties

- For any connected graph $\partial^L_n = 0$ (with multiplicity $m(0) = 1$)
- $m(\partial^L_1) \leq n - 1$, equality holds only for $K_n$
- Among trees $\partial^L_1 \geq 2n - 1$, equality holds only for $S_n$
- For $L$-spectra:

$$\mu_1(G) \geq \mu_1(G - e) \geq \mu_2(G) \geq \mu_2(G - e) \geq \cdots \geq \mu_n(G) = \mu_n(G - e) = 0$$

There is no similar result for $D^L$–spectra

- The $D^L$–spectra of $P_6$ and $C_6$ are $(21.3929, 15, 12.8532, 11, 9.7539, 0)$ and $(13, 13, 10, 9, 9, 0)$, respectively

- If $e$ is an edge in $G$ such that $G - e$ is connected, then $\partial^L_i(G - e) \geq \partial^L_i(G)$, for $i = 1, 2, \ldots, n$
- $\partial^L_i(G) \geq \partial^L_i(K_n) = n$, for $i = 1, 2, \ldots n - 1$
A connected graph $G$ is $k$–transmission regular if $t_i = k$, for $i = 1, 2, \cdots, n$

If $G$ is $k$–transmission regular with $\mathcal{D}$–spectrum $(\partial_1, \partial_2, \cdots, \partial_n)$, then $(k - \partial_n, \cdots k - \partial_1)$ is the $\mathcal{D}^L$–spectrum of $G$.

Moreover, the eigenspaces are the same.

A 7–transmission regular regular

$\mathcal{D}$–spectrum : $(7, 0, 0, -2, -2, -3)$

$\mathcal{D}^L$–spectrum : $(10, 9, 9, 7, 7, 0)$
3. Distance Laplacian

Graphs of diameter 2

Let $G$ be a graph of diameter $D = 2$, $(\mu_1, \mu_2, \ldots, \mu_n = 0)$ its $L$-spectrum and $(\partial_1, \partial_2, \ldots, \partial_n = 0)$ its $D^L$-spectrum. Then $\partial_i = 2n - \mu_{n-i}$, for $i = 1, 2, \ldots, n-1$. Moreover, the $L$–eigenspaces and $D$–eigenspaces coincide.

A 7–transmission regular regular

$L$–spectrum : $(5, 5, 3, 3, 2, 0)$

$D^L$–spectrum : $(10, 9, 9, 7, 7, 0)$
3. Distance Laplacian

**Similarities with the algebraic connectivity**

For the Laplacian $L$ [Fiedler, 1973] :
- $\mu_{n-1} = 0$ if and only if $G$ is disconnected
- The multiplicity of 0 in the $L$-spectrum of $G$ equals the number of connected components of $G$
- $\mu_{n-1}$ is called algebraic connectivity

For the distance Laplacian $D^L$ :
- $n$ is a $D^L$-eigenvalue of $G$ if and only if the complement $\overline{G}$ is disconnected
- The multiplicity of $n$ in the $D^L$-spectrum of $G$ is 1 less than the number of connected components of $\overline{G}$
3. Distance Laplacian

**Similarities with the algebraic connectivity**

**Corollaries:**
- \( \partial_1(G) \geq n \) with equality if and only if \( G \cong K_n \)
- If \( G \) is bipartite and \( n \) is a distance Laplacian eigenvalue of \( G \), then \( G \) is complete bipartite
- The star \( S_n \) is the only tree for which \( n \) is a distance Laplacian eigenvalue
- If the maximum degree \( \Delta = n - 1 \), then \( n \) is a \( D^L \)-eigenvalue with multiplicity at least \( n_\Delta \) (number of vertices of degree \( \Delta \))
4. Distance signless Laplacian

**Definition**

- The **transmission** of a vertex $i$ is the sum of all the distances from $i$ to all other vertices $t_i = \sum_{j \in V} d(i, j)$.
- The **distance Laplacian matrix** of $G$ is defined by $D^Q = Tr + D$, where $Tr$ is the diagonal matrix whose diagonal entries are the transmissions in $G$.
- The **distance Laplacian spectrum** or $D^Q$–spectrum is denoted by $(\partial^Q_1, \partial^Q_2, \ldots, \partial^Q_n)$ with $\partial^Q_1 \geq \partial^Q_2 \geq \cdots \geq \partial^Q_n$.

\[
D^Q = \begin{bmatrix}
7 & 2 & 1 & 1 & 1 & 2 \\
2 & 7 & 2 & 1 & 1 & 1 \\
1 & 2 & 7 & 2 & 1 & 1 \\
1 & 1 & 2 & 7 & 2 & 1 \\
1 & 1 & 1 & 2 & 7 & 2 \\
2 & 1 & 1 & 1 & 2 & 7 \\
\end{bmatrix}
\]

$D^Q$–spectrum: $(14, 7, 7, 5, 5, 4)$
Examples of distance signless Laplacian spectra

- For $K_n$ : $\left(2n - 2, n - 2^{(n-1)}\right)$ (also the signless Laplacian spectrum)
- For $K_n - e$ : $\left(\frac{3n - 2 \pm \sqrt{(n-2)^2 + 16}}{2}, n - 2^{(n-2)}\right)$
- For $S_n$ : $\left(\frac{5n - 8 \pm \sqrt{9n^2 - 32n + 32}}{2}, 2n - 5^{(n-2)}\right)$
- For $K_{a,b}$ : $\left(\frac{5n - 8 \pm \sqrt{9(a-b)^2 + 4ab}}{2}, 2n - a - 4^{(b-1)}, 2n - b - 4^{(a-1)}\right)$
4. Distance signless Laplacian

**Properties**

- For $Q$-spectra:

\[ q_1(G) \geq q_1(G - e) \geq q_2(G) \geq q_2(G - e) \geq \cdots \geq q_n(G) \geq q_n(G - e) \]

There is no similar result for $D^Q$-spectra.

- The $D^Q$-spectra of $P_6$ and $C_6$ are

  \((25.0838, 12.1755, 11.1743, 8.6727, 7.7418, 5.5118)\) and \((18, 9, 9, 8, 5, 5)\), respectively.

- If $e$ is an edge in $G$ such that $G - e$ is connected, then

  \[ \partial_i^Q(G - e) \geq \partial_i^Q(G), \text{ for } i = 1, 2, \ldots, n \]

- $\partial_1^Q(G) \geq \partial_1^Q(K_n) = 2n - 2$ with equality if and only if $G \cong K_n$

- $\partial_i^Q(G) \geq \partial_i^Q(K_n) = n - 2$, for $i = 2, 3, \ldots, n$

- $\partial_2^Q(G) \geq n - 2$ with equality if and only if $G \cong K_n$
4. Distance signless Laplacian

Transmission regular graphs

- $2 \text{Tr}_{\text{min}} \leq 2 \overline{\text{Tr}} \leq \partial_1^Q(G) \leq 2 \text{Tr}_{\text{max}}$ with equalities if and only if $G$ is a transmission regular graph.

- If $G$ is $k$–transmission regular with $\mathcal{D}$–spectrum $(\partial_1, \partial_2, \ldots, \partial_n)$, then $(k + \partial_1, k + \partial_2, \ldots, k + \partial_n)$ is the $\mathcal{D}^Q$–spectrum of $G$.

Moreover, the eigenspaces are the same.

A 7–transmission regular regular

$\mathcal{D}$–spectrum : $(7, 0, 0, -2, -2, -3)$

$\mathcal{D}^L$–spectrum : $(14, 7, 7, 5, 5, 4)$
**Bipartite components**

For the signless Laplacian:
- 0 is a $Q$-eigenvalue of $G$ if and only if $G$ contains a bipartite component or an isolated vertex.
- The multiplicity of 0 is equal to the number of bipartite components and isolated vertices.

For the distance signless Laplacian:
- If $n - 2$ is a $D^Q$-eigenvalue of $G$ with multiplicity $m$, then $\overline{G}$ contains at least $m$ components, each of which is bipartite or an isolated vertex.
- There exist graphs with a bipartite complement for which $n - 2$ is not a $D^Q$-eigenvalue.
4. Distance signless Laplacian

Bipartite components

$G$ (left) with $n = 5$, $\partial_5^Q \simeq 3.050286 > 3$ and $\overline{G}$ (right) bipartite
The Petersen graph and its spectra

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<th>Spectrum</th>
<th>$A$-spectrum</th>
<th>$L$-spectrum</th>
<th>$Q$-spectrum</th>
<th>$D$-spectrum</th>
<th>$D^L$-spectrum</th>
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