Generalized Inversion Sequences

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Permutations and Descents

$S_n$: set of permutations $\pi : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$

$\text{Des } \pi: \{i \in \{1, \ldots, n - 1\} | \pi(i) > \pi(i + 1)\}$ (descents)

$\text{des } \pi: |\text{Des } \pi|$, the number of descents.

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**Descent polynomial:**

$$E_n(x) = \sum_{\pi \in S_n} x^{\text{des } \pi}$$

$$E_3(x) = 1 + 4x + x^2$$
Permutations and Descents

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**Descent polynomial:**

\[ E_n(x) = \sum_{\pi \in S_n} x^{\text{des } \pi} \]

\[ E_3(x) = 1 + 4x + x^2 \]

**Eulerian polynomials:** \( E_n(x) \)
The Eulerian polynomials, $E_n(x)$

$$E_n(x) = \sum_{\pi \in S_n} x^{\text{des} \pi}$$

$$\sum_{t \geq 0} (t + 1)^n x^t = \frac{E_n(x)}{(1 - x)^{n+1}}$$

$$\sum_{n \geq 0} E_n(x) \frac{z^n}{n!} = \frac{(1 - x)}{e^{z(x-1)} - x}$$
Inversion Sequences

\[ I_n = \{(e_1, \ldots e_n) \in \mathbb{Z}^n \mid 0 \leq e_i < i\} \]

Encode permutations as inversion sequences \( \phi : S_n \to I_n \)
\( \phi(\pi) = (e_1, \ldots, e_n) \), where

\[ e_j = |\{i \mid i < j \text{ and } \pi(i) > \pi(j)\}|. \]

Example:

\( \phi(4 \ 3 \ 6 \ 5 \ 1 \ 2) = (0, 1, 0, 1, 4, 4). \)
Inversion Sequences

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What is “Asc”? 
Ascents

What is an “ascent” in an inversion sequence?

\[ e_i < e_{i+1} \]
Ascents

What is an “ascent” in an inversion sequence?

\[ e_i < e_{i+1}? \quad \text{or} \quad \frac{e_i}{i} < \frac{e_{i+1}}{i+1}? \]
Ascents

What is an “ascent” in an inversion sequence?

\[ e_i < e_{i+1} \text{? or } \frac{e_i}{i} < \frac{e_{i+1}}{i+1} \text{?} \]

Lemma

If \( 0 \leq e_j < j \) for all \( j \leq n \), then for \( 1 \leq i < n \),

\[ e_i < e_{i+1} \text{ iff } \frac{e_i}{i} < \frac{e_{i+1}}{i+1}. \]
View inversion sequences as lattice points in a (half-open) $1 \times 2 \times \cdots \times n$ box

View ascent constraints as hyperplane constraints:

$$0 < e_1 \quad \text{and} \quad \frac{e_i}{i} < \frac{e_{i+1}}{i+1}, \quad 1 \leq i < n$$

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<th>$\pi \in S_3$</th>
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**s-inversion sequences**

For any sequence $s = (s_1, s_2, \ldots, s_n)$ of positive integers:

$$I_n^{(s)} = \{(e_1, \ldots, e_n) \in \mathbb{Z}^n | 0 \leq e_i < s_i \text{ for } 1 \leq i \leq n\}.$$

(lattice points in a half-open $s_1 \times s_2 \times \cdots \times s_n$ box)

$$|I_n^{(s)}| = s_1 s_2 \cdots s_n$$
**s-inversion sequences**

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$$I_n^{(s)} = \{(e_1, \ldots, e_n) \in \mathbb{Z}^n \mid 0 \leq e_i < s_i \text{ for } 1 \leq i \leq n\}.$$ 

An \textit{ascent} of $e$ is a position $i$: $1 \leq i < n$ and

$$\frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}}.$$ 

If $e_1 > 0$ then 0 \textit{is an ascent}. 
**s-inversion sequences**

For any sequence $s = (s_1, s_2, \ldots, s_n)$ of positive integers:

$$I_n^{(s)} = \{(e_1, \ldots, e_n) \in \mathbb{Z}^n \mid 0 \leq e_i < s_i \text{ for } 1 \leq i \leq n\}.$$

Example: $(2, 4, 5) \in I_n^{(3,5,7)}$

$$\text{Asc } e = \{0, 1\}$$

$2 \notin \text{Asc } e$ since $4/5 \not< 5/7$
Ascent polynomials of $s$-inversion sequences

\[ A_n^{(s)}(x) = \sum_{e \in I_n^{(s)}} x^{\text{asc } e} \]

$(2, 4)$-inversion sequences

Ascent sets:
- $\{ \}$ yellow dot
- $\{0\}$ blue square
- $\{1\}$ red diamond
- $\{0, 1\}$ black dot

\[ A_n^{(2,4)}(x) = 1 + 6x + x^2 \]
Ascent polynomials of $s$-inversion sequences

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$$A_n^{(2,4)}(x) = 1 + 6x + x^2$$

$(3, 5)$-inversion sequences

$$A_n^{(3,5)}(x) = 1 + 10x + 4x^2$$
Call $A_n^{(s)}(x)$ the $s$-Eulerian polynomial since, when $s = (1, 2, \ldots, n)$,

$$A_n^{(s)}(x) = \sum_{e \in I_n^{(s)}} x^{\text{asc } e} = \sum_{\pi \in S_n} x^{\text{des } \pi} = E_n(x),$$

the Eulerian polynomial.

(Recall bijection $\phi : S_n \rightarrow I_n$ with $\text{Des } \pi = \text{Asc } \phi(\pi)$)
Why $s$-inversion sequences?

- Natural model for combinatorial structures
- Can prove general properties of the $s$-Eulerian polynomials
- Surprising results follow using Ehrhart theory
- Can be encoded as lecture hall partitions
- Lead to a natural refinement of the $s$-Eulerian polynomials
- Help answer questions about lecture hall partitions
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<th>$s$-Eulerian polynomial</th>
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<td>$(1, 2, 3, 4, 5, 6)$</td>
<td>$1 + 57x + 302x^2 + 302x^3 + 57x^4 + x^5$</td>
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<tr>
<td>$(2, 4, 6, 8, 10)$</td>
<td>$1 + 237x + 1682x^2 + 1682x^3 + 237x^4 + x^5$</td>
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<tr>
<td>$(6, 5, 4, 3, 2, 1)$</td>
<td>$1 + 57x + 302x^2 + 302x^3 + 57x^4 + x^5$</td>
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<tr>
<td>$(1, 1, 3, 2, 5, 3)$</td>
<td>$1 + 20x + 48x^2 + 20x^3 + x^4$</td>
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<tr>
<td>$(1, 3, 5, 7, 9, 11)$</td>
<td>$1 + 358x + 3580x^2 + 5168x^3 + 1328x^4 + 32x^5$.</td>
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<td>$(7, 2, 3, 5, 4, 6)$</td>
<td>$1 + 71x + 948x^2 + 2450x^3 + 1411x^4 + 159x^5$</td>
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Signed permutations
and
$(2, 4, 6, \ldots, 2n)$-inversion sequences

$$\left| I_{n}^{(2, 4, 6, \ldots, 2n)} \right| = 2^{n} n!$$
Signed Permutations $B_n$

$$B_n = \{ (\sigma_1, \ldots, \sigma_n) \mid \exists \pi \in S_n, \forall i \sigma_i = \pm \pi(i) \}$$

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Des $\sigma = \{ i \in \{0, \ldots, n-1\} \mid \sigma_i > \sigma_{i+1} \}$,

with the convention that $\sigma_0 = 0$.

descent polynomial:

$$1 + 6x + x^2$$
Signed Permutations $B_n$

$$B_n = \{ (\sigma_1, \ldots, \sigma_n) \mid \exists \pi \in S_n, \forall i \sigma_i = \pm \pi(i) \}$$

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Ascent sets:
- {} yellow dot
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(2, 4)-inversion sequences

Ascent polynomial:
$$1 + 6x + x^2$$

descent polynomial:
$$1 + 6x + x^2$$
Theorem (Pensyl, S 2012/13)

\[ \sum_{\sigma \in B_n} x^{\text{des } \sigma} = A_n^{(2, 4, \ldots, 2n)}(x). \]
Theorem (Pensyl, S 2012/13)

\[ \sum_{\sigma \in B_n} x^{\text{des } \sigma} = A_n^{(2,4,\ldots,2n)}(x). \]

Proof.

There is a bijection \( \Theta : B_n \rightarrow I_n^{(2,4,\ldots,2n)} \) satisfying

\[ \text{Des } \sigma = \text{Asc } \Theta(\sigma). \]
2.

\((s_1, s_2, \ldots, s_n)\)-inversion sequences vs. \((s_n, s_{n-1}, \ldots, s_1)\)-inversion sequences

Example from table:

\[ A_{6}^{(1,2,3,4,5,6)}(x) = 1 + 57x + 302x^2 + 302x^3 + 57x^4 + x^5 \]

\[ = A_{6}^{(6,5,4,3,2,1)}(x) \]
Theorem (S, Schuster 2012; Liu, Stanley 2012)

For any sequence \((s_1, s_2, \ldots, s_n)\) of positive integers,

\[
A_n^{(s_1, s_2, \ldots, s_n)}(x) = A_n^{(s_n, s_{n-1}, \ldots, s_1)}(x).
\]
Reversing $s$ preserves the ascent polynomial

$(3, 5)$-inversion sequences

- Ascent sets:
  - $\emptyset$ yellow dot
  - $\{0\}$ blue square
  - $\{1\}$ red diamond
  - $\{0, 1\}$ black dot

$1 + 10x + 4x^2$

but *not* necessarily the partition into ascent sets

$(5, 3)$-inversion sequences

$1 + 10x + 4x^2$
3.

Roots of $s$-Eulerian polynomials

Example from table:

$$A_6^{(7,2,3,5,4,6)}(x) = 1 + 71x + 948x^2 + 2450x^3 + 1411x^4 + 159x^5$$

Roots in the intervals:

$$
\left[-\frac{19}{1024}, -\frac{9}{512}\right], \left[-\frac{77}{1024}, -\frac{19}{256}\right],
\left[-\frac{423}{1024}, -\frac{211}{512}\right], \left[-\frac{1701}{1024}, -\frac{425}{256}\right],
\left[-\frac{3435}{512}, -\frac{6869}{1024}\right]
$$
Theorem (S, Visontai 2012)

For every sequence $s$ of positive integers, $A_n^{(s)}(x)$ has all real roots.
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For every sequence $s$ of positive integers, $A_n^{(s)}(x)$ has all real roots.

Corollary (Frobenius 1910; Brenti 1994)

The descent polynomials for Coxeter groups of types $A$ and $B$ have all real roots.
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The descent polynomials for Coxeter groups of types \( A \) and \( B \) have all real roots.

New: ([S, Visontai 2013]) Method can be adapted to type \( D \).
Theorem (S, Visontai 2012)

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For every sequence $s$ of positive integers, $A_n^{(s)}(x)$ has all real roots.

Corollary

For any $s$, the sequence of coefficients of the $s$-Eulerian polynomial is unimodal and log-concave.

Example $A_6^{(7,2,3,5,4,6)}(x)$:

\[ 1, 71, 948, 2450, 1411, 159, 1 \]
4. Lecture Hall Polytopes and Ehrhart Theory
Lecture hall polytopes

$s$-lecture hall polytope:

\[
\mathbf{P}_n^{(s)} = \left\{ \lambda \in \mathbb{R}^n \mid 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \leq 1 \right\}.
\]

\[
\mathbf{P}_2^{(3,5)}
\]
Lecture hall polytopes

$s$-lecture hall polytope:

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$t$-th dilation of $P_n^{(s)}$:

\[ tP_n^{(s)} = \{ t\lambda \mid \lambda \in P_n^{(s)} \}. \]
Lecture hall polytopes

$s$-lecture hall polytope:

\[ P_n^{(s)} = \left\{ \lambda \in \mathbb{R}^n \mid 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \leq 1 \right\}. \]

$t$-th dilation of \( P_n^{(s)} \):

\[ tP_n^{(s)} = \{ t\lambda \mid \lambda \in P_n^{(s)} \} \]

Ehrhart polynomial of \( P_n^{(s)} \):

\[ i(P_n^{(s)}, t) = |tP_n^{(s)} \cap \mathbb{Z}^n|. \]
Lecture hall polytopes

*s-lecture hall polytope*:

\[ P_n^{(s)} = \left\{ \lambda \in \mathbb{R}^n \mid 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \leq 1 \right\}. \]

*t-th dilation of* \( P_n^{(s)} \):

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*Ehrhart polynomial of* \( P_n^{(s)} \):

\[ i(P_n^{(s)}, t) = |tP_n^{(s)} \cap \mathbb{Z}^n|. \]
Connection between lecture hall polytopes and inversion sequences

Theorem (S, Schuster 2012)

For any sequence \( s \) of positive integers,

\[
\sum_{t \geq 0} i(P_n^{(s)}, t) \ x^t = \frac{\sum_{e \in I_n^{(s)}} x^{asc(e)}}{(1 - x)^{n+1}}.
\]
5. The sequences

\[ s = (1, 1, 3, 2, 5, 3, 7, 4, \ldots) \]

and

\[ s = (1, 4, 3, 8, 5, 12, 7, 16, \ldots) \]

Note:

\[
\left| I_{2n}^{(1, 1, 3, 2, 5, 3, \ldots, 2n-1, n)} \right| = n!(1 \cdot 3 \cdot 5 \cdot \cdots \cdot 2n - 1) = \frac{(2n)!}{2^n}
\]
Theorem \((S, \text{ Visontai 2012})\)

\[ A_{2n}^{(1,1,3,2,...,2n-1,n)} \] is the descent polynomial for permutations of the multiset \(\{1,1,2,2,...,n,n\}\).
Theorem (S, Visontai 2012)

\[ A_{2n}^{(1,1,3,2,\ldots,2n-1,n)} \] is the descent polynomial for permutations of the multiset \( \{1, 1, 2, 2, \ldots, n, n\} \).

Bijective proof?
Theorem (S, Visontai 2012)

\[ A_{2n}^{(1,1,3,2,...,2n-1,n)} \] is the descent polynomial for permutations of the multiset \( \{1, 1, 2, 2, \ldots, n, n\} \).

Conjecture (S, Visontai 2012)

\[ A_{2n}^{(1,4,3,8,...,2n-1,4n)} \] is the descent polynomial for the signed permutations of \( \{1, 1, 2, 2, \ldots, n, n\} \).
The sequences \( s = (1, k + 1, 2k + 1, 3k + 1, \ldots) \)

\[
\begin{align*}
k = 1 & : \quad (1, 2, 3, 4, 5, \ldots) \\
k = 2 & : \quad (1, 3, 5, 7, 9, \ldots)
\end{align*}
\]
The sequences $s = (1, k + 1, 2k + 1, 3k + 1, \ldots)$

$k = 1 : \quad (1, 2, 3, 4, 5, \ldots)$

$k = 2 : \quad (1, 3, 5, 7, 9, \ldots)$

Let:

$$I_{n,k} = I_{n}^{(1,k+1,2k+1,\ldots,(n-1)k+1)}$$

$$A_{n,k}(x) = A_{n}^{(1,k+1,2k+1,\ldots,(n-1)k+1)}(x)$$

Recall $A_{n,1}(x) = E_{n}(x)$. 
The 1/$k$-Eulerian polynomials

Theorem (S, Viswanathan 2012)

For positive integer $k$,

\[ A_{n,k}(x) = \sum_{e \in I_{n,k}} x^{\text{asc } e} \]

\[ \sum_{t \geq 0} \left( \frac{t - 1 + \frac{1}{k}}{t} \right) (kt + 1)^n x^t = \frac{A_{n,k}(x)}{(1 - x)^{n + \frac{1}{k}}} \]

\[ \sum_{n \geq 0} A_{n,k}(x) \frac{z^n}{n!} = \left( \frac{1 - x}{e^{kz(x-1)} - x} \right)^{\frac{1}{k}} \]
Theorem (S, Viswanathan 2012)

\[
\sum_{e \in I_{n,k}} x^{\text{asc } e} = \sum_{\pi \in S_n} x^{\text{exc } \pi} k^{n - \#\text{cyc } \pi},
\]

where

\[
\text{exc } \pi = |\{i \mid \pi(i) > i\}|
\]

and \(\#\text{cyc } \pi\) is the number of cycles in the disjoint cycle representation of \(\pi\).
Theorem (S, Viswanathan 2012)

\[
\sum_{e \in I_{n,k}} x^{\text{asc } e} = \sum_{\pi \in S_n} x^{\text{exc } \pi} k^{n-\#\text{cyc } \pi},
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\text{exc } \pi = |\{i \mid \pi(i) > i\}|
\]

and \#\text{cyc } \pi is the number of cycles in the disjoint cycle representation of \( \pi \).

Combinatorial proof?
7.

Lecture Hall Partitions

\[ L_n = \left\{ \lambda \in \mathbb{Z}^n \mid \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \cdots \leq \frac{\lambda_n}{n} \right\} \]
7.

Lecture Hall Partitions

\[ L_n = \left\{ \lambda \in \mathbb{Z}^n \mid \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \cdots \leq \frac{\lambda_n}{n} \right\} \]
s-lecture hall partitions

\[ L_n^{(s)} = \left\{ \lambda \in \mathbb{Z}^n \mid \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \right\} \]

Theorem (Bousquet-Mélou, Eriksson 1997)

For \( s = (1, 2, \ldots, n) \),

\[
\sum_{\lambda \in L_n^{(s)}} q^{\left| \lambda \right|} = \frac{1}{(1 - q)(1 - q^3) \cdots (1 - q^{2n-1})},
\]

where \( \left| \lambda \right| = \lambda_1 + \cdots + \lambda_n \).
s-lecture hall partitions

\[ L_n^{(s)} = \left\{ \lambda \in \mathbb{Z}^n | \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \right\} \]

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\]

where \( |\lambda| = \lambda_1 + \cdots + \lambda_n \).

(What other sequences \( s \) give rise to nice generating functions?)
8.

Fundamental Lecture Hall Parallelepiped

\[ s = (3, 5) \]

\[ s = (2, 4, 6) \]
Fundamental (half-open) $s$-lecture hall parallelepiped:

$$
\Pi_n^{(s)} = \left\{ \sum_{i=1}^{n} c_i w_i \mid 0 \leq c_i < 1 \right\},
$$

where $w_i = [0, \ldots, 0, s_i, s_{i+1}, \ldots s_n]$.

**Theorem (Liu, Stanley 2012)**

*There is a bijection between $I_n^{(s)}$ and $\Pi_n^{(s)} \cap \mathbb{Z}^n$.***
$I_2^{(3,5)} \rightarrow \Pi_2^{(3,5)} \cap \mathbb{Z}^2$
9.

Inflated $s$-Eulerian polynomials

$$\sum_{\lambda \in \Pi_n^{(s)}} x^{\lambda_n}$$
Define the *inflated s-Eulerian polynomial* by

\[ Q_n^{(s)}(x) = \sum_{\lambda \in \Pi_n^{(s)}} x^{\lambda_n}. \]

**Theorem (Pensyl,S 2013)**

*For any sequence \( s \) of positive integers,*

\[ Q_n^{(s)}(x) = \sum_{e \in I_n^{(s)}} x^{s_{\text{asc}} e - e_n} \]
For \( s = (1, 2, \ldots, n) \), the coefficient sequence of \( Q_n^{(s)} \) gives an interesting refinement of the Eulerian numbers:

Coefficient sequence:

\[ 1, 1, 2, 4, 4, 4, 4, 2, 1, 1 \]
For $s = (1, 3, \ldots, 2n - 1)$, the coefficient sequence of $Q_n^{(s)}(x)$ is symmetric (but not the coefficient sequence of $A_n^{(s)}(x)$.)

Coefficient sequence:

1, 1, 2, 4, 4, 6, 9, 10, 10, 11, 10, 10, 9, 6, 4, 4, 2, 1, 1
For $s = (1, 1, 2, 3, 5, 8, \ldots)$, the coefficient sequence of $Q_n^{(s)}$ is not symmetric for $n \geq 5$.

Coefficient sequence:

1, 1, 2, 2, 4, 4, 4, 4, 4, 2, 1, 1,
10.

Gorenstein cones and self-reciprocal generating functions

*Self-reciprocal:*

satisfies \( f(q) = q^b f(1/q) \) for some nonnegative integer \( b \)

Examples:

\[
1 + x + 2x^2 + 4x^3 + 4x^4 + 4x^5 + 4x^6 + 2x^7 + x^8 + x^9
\]

\[
\frac{1}{(1 - q)(1 - q^3)(1 - q^5)}
\]
A pointed rational cone $C \subseteq \mathbb{R}^n$ is Gorenstein if there exists a point $c$ in the interior $C^0$ of $C$ such that $C^0 \cap \mathbb{Z}^n = c + (C \cap \mathbb{Z}^n)$.

Theorem (Special case of a result due to Stanley 1978)

The $s$-lecture hall cone is Gorenstein if and only if $Q_n^{(s)}(x)$ is self-reciprocal; also, if and only if the following is self reciprocal:

$$f_n^{(s)}(q) = \sum_{\lambda \in L_n^{(s)}} q^{||\lambda||}$$
Theorem (Bousquet-Mélou, Eriksson 1997; Beck, Braun, Köppe, S, Zafeirakopoulos 2012)

The s-lecture hall cone is Gorenstein if and only if there exists \( c \in \mathbb{Z}^n \) satisfying

\[
c_j s_{j-1} = c_{j-1} s_j + \gcd(s_j, s_{j-1})
\]

for \( j > 1 \) with \( c_1 = 1 \).
Theorem (BBKSZ)

[Beck, Braun, Köppe, S, Zafeirakopoulos 2012]

Let $s$ be a sequence of positive integers defined by

$$s_n = \ell s_{n-1} + ms_{n-2}, (*)$$

with $s_0 = 0, s_1 = 1$. Then the $s$-lecture hall cone in Gorenstein for all $n$ if and only if $m = -1$. 
Theorem (BBKSZ)

[Beck, Braun, Köppe, S, Zafeirakopoulos 2012]

Let $s$ be a sequence of positive integers defined by

$$s_n = \ell s_{n-1} + ms_{n-2}, (*)$$

with $s_0 = 0, s_1 = 1$. Then the $s$-lecture hall cone in Gorenstein for all $n$ if and only if $m = -1$.

Sequences $(*)$ with $m = -1$ are called $\ell$-sequences.
\( \ell \)-sequences

\[
s_n = \ell s_{n-1} - s_{n-2},
\]

with \( s_0 = 1, \ s_1 = 1. \)

\( \ell = 2 \)

1, 2, 3, 4, 5, 6, 7, 8, 9, ...

\( \ell = 3 \)

1, 3, 8, 21, 55, 144, 377, 987, 2584, ...
Theorem (Bousquet-Mélou, Eriksson 1997)

If $s$ is an $\ell$-sequence,

$$
\sum_{\lambda \in L_n^{(s)}} q^{\lambda} = \frac{1}{(1 - q)(1 - q^{s_1+s_2})(1 - q^{s_2+s_3}) \cdots (1 - q^{s_{n-1}+s_n})}.
$$

Conversely, by the BBKSZ Theorem, for a sequence of the form (*) unless $s$ is an $\ell$-sequence, the $s$-lecture hall partitions cannot, for all $n$, have a generating function of the form

$$
\frac{1}{(1 - q^{c_1})(1 - q^{c_2}) \cdots (1 - q^{c_n})}.
$$
Question

What combinatorial family is being represented by the $s$-inversion sequences when $s$ is an $\ell$-sequence?

(When $\ell = 2$, the answer is permutations.)
References


10. The Eulerian polynomials of type $D$ have only real roots, C. D. Savage and M. Visontai, FPSAC 2013, Paris.


Thank you!