

Non-trivial Decidable Nested Recurrence Relations

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June 5, 2013

Overview

Background

Decidable Nested Recurrences - Bounded Example

Decidable Nested Recurrences - Unbounded Example

Open Problems

Nested Recurrence Relations

- ▶ Expression of the form $\dots F(\dots F(\dots)\dots)\dots$ appears in the recursive definition for F .
- ▶ In other words, previous values are used to determine how far to look back for values, which are in turn used to generate the new number in the sequence.
- ▶ Discussed in “Godel, Escher, Bach: an Eternal Golden Braid” by Douglas R. Hofstadter, where Q sequence, in particular, is introduced.
- ▶ Also studied by Golomb, Ruskey, Tanny, etc.
- ▶ Some display chaotic behaviour, highly dependent on initial conditions.

Decidability of Nested Recurrence Relations

- ▶ Celaya and Ruskey, in their 2012 paper, study nested recurrence relations from a computational perspective.
- ▶ The idea is to consider a recurrence relation, such as $G(n) = G(n - G(n - 1))$, as an abstract computing machine.
- ▶ The recurrence takes in as input a set of initial conditions, such as $F(1) = 1$ and $F(2) = 2$.
- ▶ The recurrence then produces an output in the form of a sequence of numbers, in the above example $1, 2, 1, 1, 1, \dots$

Decision Problems and Nested Recurrence Relations

- ▶ Similar to tag systems' halting condition, a nested recurrence relation can reach points in the sequence where it can no longer generate a new number based on the recurrence.
- ▶ Simple example: let $G(n) = G(n - G(n - 1))$ and $G(1) = 2$, then $G(2) = G(2 - 2) = G(0)$ which is undefined.
- ▶ This is referred to as being “well-defined”, and also as “living or dying”.
- ▶ We can formulate decision problems using this condition.

Undecidable Example

- ▶ Celaya and Ruskey provide an example of a universal nested recurrence relation in their 2012 paper.
- ▶ The recurrence is

$$A(n) = A(n - 4 - A(A(n - 4))) + 4A(A(n - 4)) + A(2A(n - 4 - A(n - 2)) + A(n - 2)).$$

- ▶ Universality is proved by showing that the recurrence can simulate reverse tag systems, which can in turn simulate tag systems.
- ▶ Question of decidable examples is brought up.

Two observations about $G(n - G(n - 1))$

- ▶ **Finiteness:** Only a finite set of values are ever part of the sequence, namely the values given in the initial conditions.
- ▶ **Boundedness:** The furthest the recurrence ever goes back is determined by the largest number in the initial conditions.
- ▶ Example: In the above graph, the initial conditions are 9, 3, 6, 5, 2, 2, 8, 5, 8, 6 so only values in $\{2, 3, 5, 6, 8, 9\}$ are going to be in the sequence. Similarly, the furthest the recurrence rule will ever look back is $K = 9$ numbers back.

Simplified Decidability Result for $G(n - G(n - 1))$

- ▶ Let K be the largest number in the initial conditions, and R the set of numbers in the initial condition.
- ▶ Construct a graph with vertices $R^K \cup \{\perp\}$. Connect (s_1, s_2, \dots, s_K) to $(s_2, s_3, \dots, s_K, s_{K+1-s_K})$ if $s_K \leq K$ and label the edge with s_{K+1-s_K} , otherwise connect it to \perp and label the edge with \perp .
- ▶ Connect \perp to \perp and label the edge \perp .

Simplified Decidability Result - Continued

- ▶ Starting with a set of K initial conditions, $G(1) = s_1, \dots, G(K) = s_K$, find the vertex (s_1, \dots, s_K) on the graph, and follow its infinite traversal.
- ▶ The edge labels then give the sequence G . If the sequence becomes undefined at any point, then we get \perp at that point onwards.
- ▶ Since the graph is finite by construction, either \perp or another cycle is reached. This implies that either the sequence dies, or it becomes periodic at some point.

Example Functional Di-Graph

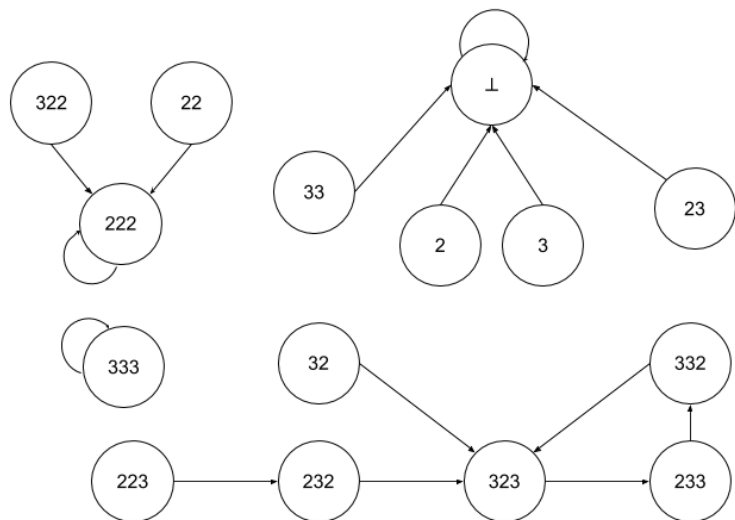


Figure : Example directed graph constructed for initial conditions 3, 2.

Generalization

Generalizes to any nested recurrence relation that satisfies the **finiteness** and **boundedness** conditions. Examples:

- ▶ $G(n) = G(n - G(n - 1) - G(n - 2))$
- ▶ $G(n) = G(n - 2G(n - 1) + 3G(n - 2))$
- ▶ $G(n) = G(n - G(G(n - 1) + G(n - 2)))$
- ▶ etc.

Nested Recurrences as Discrete Dynamical Systems

It is to interesting point out that some recurrences that satisfy the **finiteness** and **boundedness** conditions can be embedded in discrete dynamical systems on reals. A brief construction of the dynamical system corresponding to $G(n) = G(n - G(n - 1))$ is given here:

- ▶ Assume that b is a number that is larger than the largest given initial condition.
- ▶ Represent initial segments of the sequence in reverse as base b decimals. For example, letting $b = 10$, represent $(1, 3, 2, 4, 2, 5)$ with $x = 0.524231$.
- ▶ Then $G(n - k)$ corresponds to $\lfloor b^k x \rfloor$.
- ▶ After some algebra and simplification we get the following function acting on $[0, 1]$ as the dynamical system:

$$\Xi(x) = \frac{1}{b} \lfloor b^{\lfloor bx \rfloor} x \rfloor - \lfloor b^{\lfloor bx \rfloor - 1} x \rfloor + \frac{x}{b}$$

Graph of $\Xi(x)$

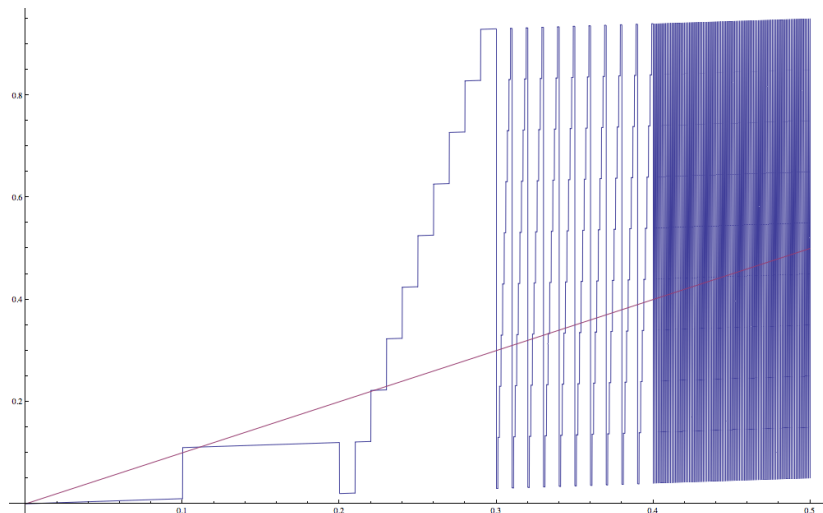


Figure : Graph of $\Xi(x)$ with $b = 10$ and the line $y = x$.

Non-homogenous case: $G(n) = 1 + G(n - G(n - 1))$

Now let us consider $G(n) = 1 + G(n - G(n - 1))$.

- ▶ Sequences given by this recurrence relation are unbounded provided they are well-defined.
- ▶ Therefore further and further previous numbers are needed to generate the next number.
- ▶ Meaning both above observations fail.
- ▶ Need a new approach to prove decidability.
- ▶ However, looking at the graphs, an observable pattern definitely exists. See below for a few examples.

$G(n) = 1 + G(n - G(n - 1))$ Example 1

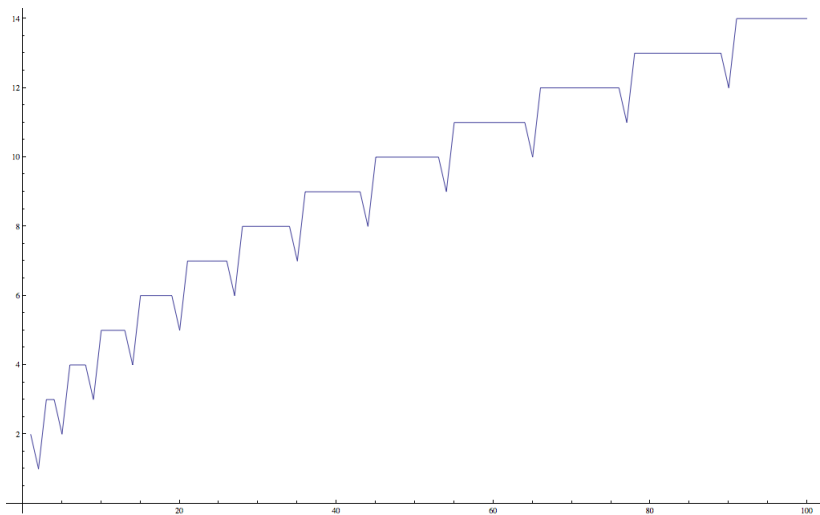


Figure : Graph of $(n) = 1 + G(n - G(n - 1))$ with initial conditions $(2, 1, 3)$.

$G(n) = 1 + G(n - G(n - 1))$ Example 2

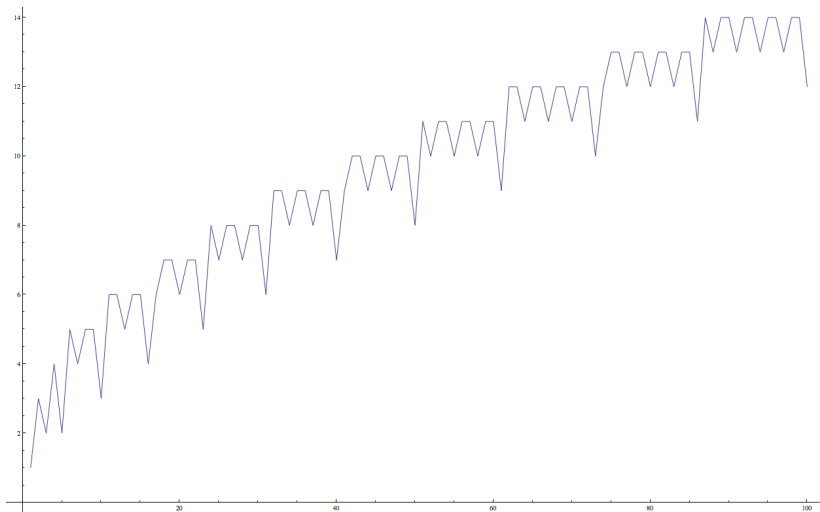


Figure : Graph of $(n) = 1 + G(n - G(n - 1))$ with initial conditions $(1, 3, 2, 4)$.

$G(n) = 1 + G(n - G(n - 1))$ Example 3

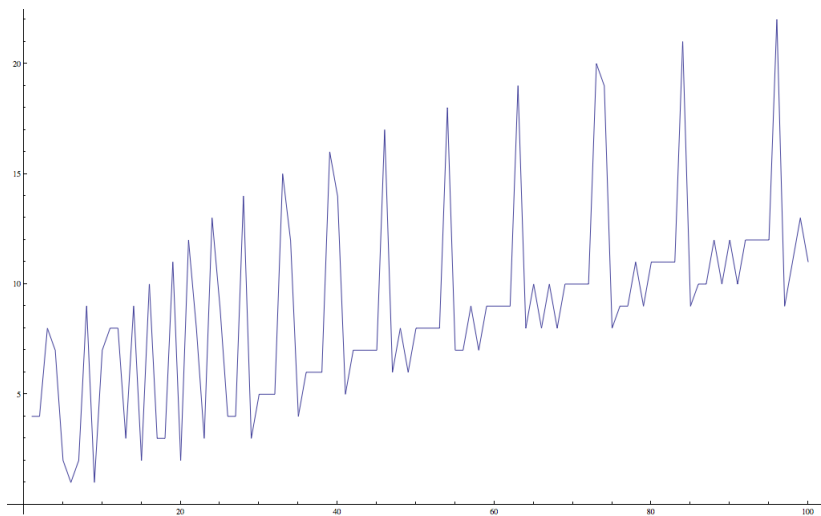


Figure : Graph of $(n) = 1 + G(n - G(n - 1))$ with initial conditions $(4, 4, 8, 7, 2, 1, 2, 9, 1, 7)$.

Sketch of the Decidability Proof

Decidability can be proved by showing the following:

1. Let $M(n) = \max_{i=1}^n G(i)$ and K the number of initial conditions. Then $M(n)$ is slow-growing for $n \geq K$.
2. If for some n_0 we have $M(n_0) \leq n_0$ then the sequence lives.
3. $M(n)$ does not grow in two consecutive positions.
4. Therefore $G(n) \leq C + \frac{n}{2}$ for $n > N$ for some constants N, C both of which are computable from the initial conditions.
5. This means n_0 required in condition 2 above can be found to be less than $2M(K)$.
6. End result: if the sequence dies, it dies before $2M(K)$ numbers.

Open Problems

- ▶ $M(n) = \Theta(\sqrt{n})$ provided the sequence lives.
- ▶ Complete characterization of sequences generated by $G(n) = 1 + G(n - G(n - 1))$. See next slide.
- ▶ Generalize to $G(n) = v + G(n - G(n - 1))$ for $v \geq 2$.
- ▶ Is $Q(n) = Q(n - Q(n - 1)) + Q(n - Q(n - 2))$ decidable or undecidable?
- ▶ Can the decision problem for $Q(n) = Q(n - Q(n - 1)) + Q(n - Q(n - 2))$ be reduced to that of other similar “meta-Fibonacci” recurrences, such as $C(n) = C(n - C(n - 1)) + C(n - 1 - C(n - 2))$?

Conjecture About $G(n) = 1 + G(n - G(n - 1))$

- ▶ Let A_k be the sequence of indices where $M(n)$ grows.
- ▶ Define generations g_k for $k \geq 0$ by letting g_k be the word $(G(A_{k+1}), G(A_{k+1} + 1), G(A_{k+1} + 2), \dots, G(A_{k+2} - 1))$.
- ▶ Then assuming $G(n)$ lives, as an infinite word we have $G = g_0 g_1 g_2 g_3 g_4 \dots$
- ▶ Let $\hat{g}_k = g_k - k$ where subtraction is done component wise.
- ▶ Conjecture: There exists K and $p \geq 1$ such that for $i > K$ we have

$$\hat{g}_{ip+r} = x_r y^{i-1} z_r$$

for $0 \leq r < p$, where $|y| = p$.

Example

- ▶ Take initial conditions $(1, 3, 2, 4)$. Graph is given in Example 2 in previous slides. Omitting brackets and commas we have:

$$\hat{g}_0 = 13242$$

$$\hat{g}_1 = 43442$$

$$\hat{g}_2 = 4434424$$

$$\hat{g}_3 = 443442$$

$$\hat{g}_4 = 43443442$$

...

- ▶ We get $p = 3$, $y = 443$ and the following:

$$\hat{g}_{3k} = 443(443)^{k-1}442$$

$$\hat{g}_{3k+1} = 43443(443)^{k-1}442$$

$$\hat{g}_{3k+2} = 443(443)^{k-1}4424$$

The End