# Nested recursions, simultaneous parameters, and tree superposition

Mustazee Rahman

University of Toronto

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# Our focus

Tree based combinatorial interpretations for many nested recursions of the form:

$$R(n) = \sum_{i=1}^{k} R(n - a_i - \sum_{j=1}^{p} R(n - b_{ij}))\,.$$

The initial conditions are specified by the associated tree structure.

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The initial conditions are specified by the associated tree structure.

- Conolly:
  $C(n) = C(n - C(n-1)) + C(n - 1 - C(n-2))$; $C(1) = 1, C(2) = 2$.
- H-recursion:
  $H(n) = H(n - H(n-1)) + H(n - 2 - H(n-3))$; $H(1) = H(2) = 1$.

# Tree based interpretation of Conolly

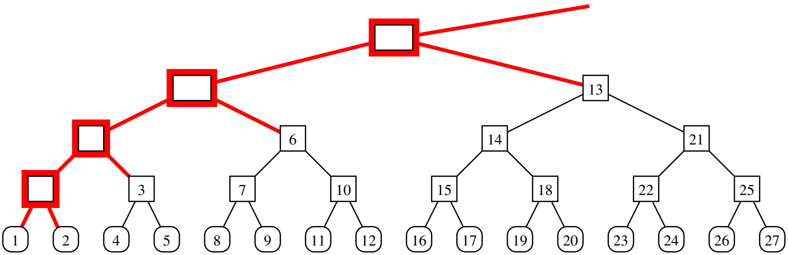


Figure : $T_C$ - An infinite binary tree labeled in preorder.

$C(n) = \#$ of leaves in $T_C$ until node $n$. E.g. $C(5) = 4$ and $C(7) = 4$.
Jackson, Ruskey (2006):

$$C(n) = C(n - C(n-1)) + C(n - 1 - C(n-2)).$$

# The simultaneous parameter $s$

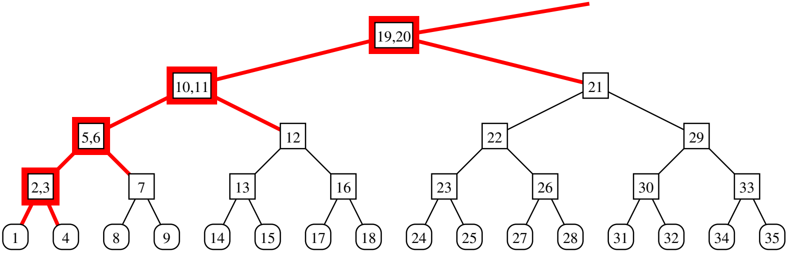


Figure : The tree $T_s$ where $s = 2$.

Let $T_s$ be the labeled infinite binary Conolly tree $T_C$ with $s$ labels in each supernode.

$C_s(n) = \#$ of leaves until label $n$ of $T_{C_s}$. E.g. $C_2(5) = 2$.

Jackson, Ruskey (2006):

$$C_s(n) = C_s(n - s - C_s(n-1)) + C_s(n - s - 1 - C_s(n-2)).$$

# The simultaneous parameter $j$

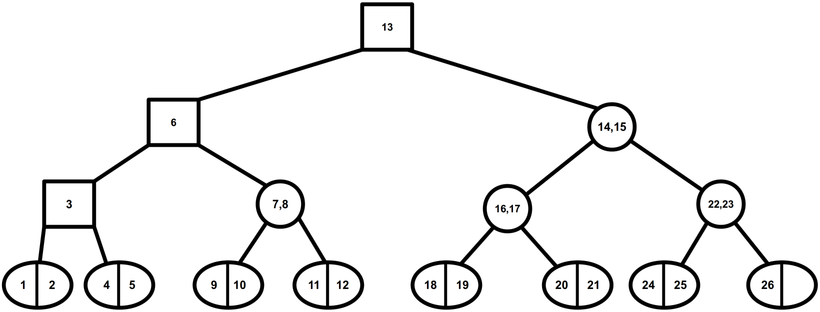


Figure : The tree $T_{s,j}$ where $s = 1$ and $j = 2$.

Let $T_{s,j}$ be the labeled binary tree $T_s$ with $j$ labels in each regular node.
$C_{s,j}(n) = \#$ of leaf labels until label $n$ of $T_{s,j}$. E.g. $C_{1,2}(5) = 4$.
Isgur, Reiss, Tanny (2009):

$$C_{s,j}(n) = C_{s,j}(n - s - C_{s,j}(n - j)) + C_{s,j}(n - s - j - C_{s,j}(n - 2j)) .$$

# Simultaneous parameters $s$ and $j$ for the $H$ recursion

The parameters $s$ and $j$ can be introduced into the $H$ recursion to yield

$$H_{s,j}(n) = H_{s,j}(n - s - H_{s,j}(n - j)) + H_{s,j}(n - s - 2j - H_{s,j}(n - 3j)) \,.$$

Isgur, Reiss, and Tanny provide a tree based interpretation for this recursion in terms of an infinite labeled tree $T_{H_{s,j}}$.

As with the case for $C_{s,j}$ the parameter $s$ is the number of labels within the super nodes of the associated tree, and $j$ is the number of labels in the corresponding regular nodes.

# The simultaneous parameter $m$: interpolating between the trees $T_{s,j}$ and $T_{H_{s,j}}$.

The trees $T_{s,j}$ and $T_{H_{s,j}}$ turn out to be the endpoints of a one parameter family of infinite labeled trees $T_{s,j,m}$ for $0 \leq m \leq j$.

The tree $T_{s,j,m}$ can be used to derive a solution to the recursion

$$C_{s,j,m}(n) = C_{s,j,m}(n-s-C_{s,j,m}(n-j)) + C_{s,j,m}(n-s-j-m-C_{s,j,m}(n-2j-m)).$$

The parameter $m$ determines the number of labels inserted within the regular nodes.

# Construction of $T_{s,j,m}$

Start with the Conolly tree $T_C = T_{0,1}$ and do as follows:

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Start with the Conolly tree $T_C = T_{0,1}$ and do as follows:

- Add $s$ labels to each supernode and $j - m$ labels to each regular node that is not a leaf.
- Subdivide each leaf node into $j$ cells. Insert 1 label in the first $j - 1$ cells and $1 + m$ labels in the last cell. In total a leaf contains $j + m$ labels distributed within $j$ cells.
- Enumerate the labels in preorder.

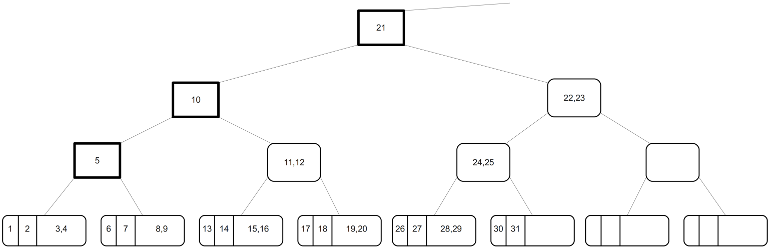


Figure : The tree $T_{1,3,1}$.

## Theorem

*Let $C_{s,j,m}(n)$ be the number of nonempty cells in $T_{s,j,m}$ up to (and including) label n. Then for $n > 5j + 3m + 2s$, $C_{s,j,m}$ satisfies*

$$C_{s,j,m}(n) = C_{s,j,m}(n-s-C_{s,j,m}(n-j)) + C_{s,j,m}(n-s-j-m-C_{s,j,m}(n-2j-m))$$

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The proof uses a basic *pruning operation* on $T = T_{s,j,m}$ to provide a bijection between the L.H.S. and R.H.S. above.

**initial correction step:** Remove $s$ labels from the first supernode and convert it into a regular node.

**deletion step:** Delete the largest label from every cell of $T$.

**lifting step:** Move any remaining labels in a cell into its parent node. After this step all cells are empty and the nodes at the penultimate level contain $j + m$ labels each.

**relabelling step:** Delete all the empty leaves and subdivide the labels of the penultimate level nodes into $j$ cells as in $T$. These nodes are now leaves of the pruned tree.

# Sketch of proof

- The pruned tree $\mathcal{P}T$ is the same as $T$. This invariance is crucial in order to establish the bijection.
- The proof uses a slightly modified pruning operation to the subtree $T(n)$ consisting of all labels of $T$ up to label $n$.
- The pruning operation is used to show that the term $C_{s,j,m}(n - s - C_{s,j,m}(n - j))$ counts all nonempty cells of $T(n)$ that are within left leaves. Similarly, $C_{s,j,m}(n - s - j - m - C_{s,j,m}(n - 2j - m))$ counts the number of nonempty cells that are in the right leaves of $T(n)$.

# Extension to order $p$

The trees $T_{s,j,m}$ are part of a larger family of tress $T_{s,j,m,p}$, $p \geq 1$ and $0 \leq m \leq (2p-1)j$ with $T_{s,j,m} = T_{s,j,m,1}$.

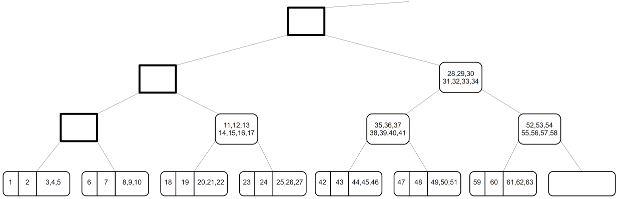


Figure : The tree $T_{0,3,2,2}$.

$T_{s,j,m,p}$ has the same structure as $T_{s,j,m}$, however, it is invariant under a more general pruning operation. The pruning operation has $p$ deletion stages that allows us to express the cell counting function of $T_{s,j,m,p}$ as the solution to an order $p$ nested recursion.

## Theorem

*The cell counting function* $C(n) = C_{s,j,m,p}(n)$ *associated to* $T_{s,j,m,p}$ *satisfies*

$$\begin{aligned} C(n) &= C(n - s - \sum_{i=1}^{p} C(n - (2i-1)j)) \\ &+ C(n - s - j - m - \sum_{i=1}^{p} C(n - j - m - (2i-1)j)) \end{aligned}$$

*for n sufficiently large.*

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*for $n$ sufficiently large.*

When $m = 0$, $C_{s,j,m,p}$ is the order $p$ analogue of $C_{s,j}$.

When $m = (2p-1)j$, $C_{s,j,m,p}$ is the order $p$ analogue of $H_{s,j}$.

In particular, $C_{0,1,2p-1,p}(n) = \lceil \frac{n}{2p} \rceil$.

# Tree superposition and $(\alpha, \beta)$-Conolly recursions

The $(\alpha, \beta)$-Conolly recursions are

$$\begin{aligned} R_{\alpha,\beta}(n) &= R_{\alpha,\beta}\Big(n - \sum_{i=1}^{p} R_{\alpha,\beta}(n-(2i-1)j)\Big) \\ &+ R_{\alpha,\beta}\Big(n-(\alpha+\beta) - \sum_{i=1}^{p} R_{\alpha,\beta}(n-(\alpha+\beta)-(2i-1)j)\Big) \end{aligned}$$

with $\alpha$ even, $\beta \geq 0$, $p = \alpha/2 + \beta$, and $\alpha + \beta \geq 1$.

Erickson, Isgur, Jackson, Ruskey, and Tanny (2011) provide a tree based solution to $R_{\alpha,\beta}$ in terms of the cell counting function of a tree $T_{\alpha,\beta}$.

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Erickson, Isgur, Jackson, Ruskey, and Tanny (2011) provide a tree based solution to $R_{\alpha,\beta}$ in terms of the cell counting function of a tree $T_{\alpha,\beta}$.

The key feature of their solution is that the frequency sequence $\phi_{R_{\alpha,\beta}} = \alpha/2\phi_H + \beta\phi_C$.

This holds because $T_{\alpha,\beta}$ is the *superposition* of copies of $T_C$ and $T_H$: $T_{\alpha,\beta} = \alpha/2T_H + \beta T_C$.

# $C_{s,j,m,p}$ extends $R_{\alpha,\beta}$

The family $C_{s,j,m,p}$ contains $R_{\alpha,\beta}$ by taking $j = 1$, $m = \alpha + \beta - 1$, and $p = \alpha/2 + \beta$.
Also, for arbitrary $j$ and $m = (\alpha + \beta - 1)j$, the sequence $C_{0,j,(\alpha+\beta-1)j,\alpha/2+\beta}$ satisfies the $(\alpha, \beta)$-Conolly recursion with simultaneous parameter $j$:

$$\begin{aligned} R_{\alpha,\beta,j}(n) &= R_{\alpha,\beta,j}(n - \sum_{i=1}^{p} R_{\alpha,\beta,j}(n - (2i-1)j)) \\ + & \quad R_{\alpha,\beta,j}(n - (\alpha+\beta)j - \sum_{i=1}^{p} R_{\alpha,\beta,j}(n - (\alpha+\beta)j - (2i-1)j)). \end{aligned}$$

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Also, for arbitrary $j$ and $m = (\alpha + \beta - 1)j$, the sequence $C_{0,j,(\alpha+\beta-1)j,\alpha/2+\beta}$ satisfies the $(\alpha, \beta)$-Conolly recursion with simultaneous parameter $j$:

$$
\begin{aligned}
R_{\alpha,\beta,j}(n) &= R_{\alpha,\beta,j}(n - \sum_{i=1}^{p} R_{\alpha,\beta,j}(n - (2i-1)j)) \\
+ \quad & R_{\alpha,\beta,j}(n - (\alpha+\beta)j - \sum_{i=1}^{p} R_{\alpha,\beta,j}(n - (\alpha+\beta)j - (2i-1)j)).
\end{aligned}
$$

However, $\phi_{C_{0,j,(\alpha+\beta-1)j,\alpha/2+\beta}}$ is *not a linear combination* of $\phi_{C_{0,j}}$ and $\phi_{H_{0,j}}$.

The corresponding $T_{0,j,m,p}$ is not a superposition of $T_{C_j}$ and $T_{H_j}$.

# Superposition of $T_{C_{0,j}}$ and $T_{H_{0,j}}$

## Theorem

*Let $j \geq 1, \gamma, \delta \geq 0$ such that $p = \gamma + \delta \geq 1$. Then the cell counting function $C(n)$ of the superposed tree $\gamma T_{H_{0,j}} + \delta T_{C_{0,j}}$ satisfies the following nested recursions for $n \geq (8\gamma + 5\delta)j$:*

$$
\begin{aligned}
C(n) &= C(n - \sum_{i=1}^{p} C(n - (2i-1) - p(j-1))) \\
&+ C(n - (p+\gamma)j - \sum_{i=1}^{p} C(n - (p+\gamma)j - (2i-1) - p(j-1)).
\end{aligned}
$$

*Consequently, the frequency sequence $\phi_C = \gamma\phi_{H_{0,j}} + \delta\phi_{C_{0,j}}$.*

# The End

Thanks for your interest!