

Avoiding Circular Repetitions

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Avoidability

- The seminal paper of Axel Thue (1906).
- Thue asked the existence of an infinite word over the alphabet $\{a, b, c\}$ that avoids *squares*, i.e., factors of the form xx .
- Constructed a *squarefree* word by iterating the following *morphism*

$$a \rightarrow abcab$$

$$b \rightarrow acabcb$$

$$c \rightarrow acbcacb$$

- Iterating it gives us

$$a \rightarrow abcab \rightarrow abcabacabcbacbcacbabcabacabcb \rightarrow \dots$$

- Ternary alphabet is the smallest with an infinite squarefree word.

Various themes

- Is the *pattern* $ABCBABC$ avoidable?
- Is it 3-avoidable?
- Smallest n for which P is n -avoidable?
- Decidability
- Over *circular words* (necklaces)
- **Smallest avoidable exponent (repetition threshold)**

Repetitions

- k -power: $x^k = \overbrace{xx \cdots x}^k$
- x^2 is 3-avoidable but not 2-avoidable (Thue).
- α -power: $y = x^{[\alpha]}x'$ such that $\frac{|y|}{|x|} = \alpha$. We then write

$$y = x^\alpha.$$

Examples

- hotshots = (hots)²
- alfalfa = (alf)²a = (alf) ^{$\frac{7}{3}$}

Definition

- w is α -power-free if none of its factors is a β -power for any $\beta \geq \alpha$.
- w is α^+ -power-free if none of its factors is a β -power for any $\beta > \alpha$.

Conjugation

- Two words x, y are *conjugate* if one is a cyclic shift of the other.
- That is if there exist words u, v such that $x = uv$ and $y = vu$.
- 'bookcase' and 'casebook'.
- One simple observation is that all conjugates of a k -power are k -powers.

$$abcabc = (abc)^2$$

$$cabcab = (cab)^2$$

Circular repetition

Observation

w is α -power-free if for every factor x

$$w = \boxed{} \boxed{x} \boxed{}$$

x is α -power-free

Definition

w is *circularly* α -power-free if for every factor x

$$w = \boxed{} \boxed{x} \boxed{}$$

x and all its conjugates are α -power-free.

Circular repetition: example

Example

$w = \text{dividing}$

$x = \text{dividi}$

a conjugate of x is

vididi

which has a $\frac{5}{2}$ -power: $\text{ididi} = (\text{id})^{\frac{5}{2}}$

- So w is not circularly $\frac{5}{2}$ -power-free.
- In fact, w is circularly $(\frac{5}{2})^+$ -power-free.

Circular repetition

- w is circularly α -power-free if for every pair of factors x and y

$$w = \boxed{\quad} \boxed{x} \boxed{\quad} \boxed{y} \boxed{\quad}$$

yx is α -power-free.

- (x, y) is a circular α -power if yx is α -power.

Example

$$w = \begin{array}{|c|c|c|c|c|c|c|c|} \hline d & i & v & i & d & i & n & g \\ \hline \end{array}$$

$\underbrace{\hspace{2em}}_x \qquad \underbrace{\hspace{4em}}_y$

$$yx = \text{ididi} = (\text{id})^{\frac{5}{2}}$$

Hence (x, y) is a circular $\frac{5}{2}$ -power.

Repetition threshold

Definition

The *repetition threshold*, $RT(n)$, is the smallest α for which there exists an infinite α^+ -power-free word over Σ_n .

Example

Dejean proved $RT(3) = \frac{7}{4}$ by proving

- 1 there are only finitely many $\frac{7}{4}$ -power-free words.
- 2 there are infinite $(\frac{7}{4})^+$ -power-free words.

$$\nu(a) = abcacbcabcbacbcacba,$$

$$\nu(b) = bcabacabcacbacabacb,$$

$$\nu(c) = cabcbabcabacbabcbac.$$

$$\nu^\omega(a) = a \overbrace{bcacbc} bcbacbcacba \dots$$

Dejean's conjecture

Dejean's conjecture

Thue, Dejean, Pansiot, Moulin Ollagnier, Carpi, Currie, Mohammad-Noori, Rampersad, and Rao:

$$RT(n) = \begin{cases} \frac{7}{4}, & \text{if } n = 3; \\ \frac{7}{5}, & \text{if } n = 4; \\ \frac{n}{n-1}, & \text{if } n \neq 3, 4. \end{cases}$$

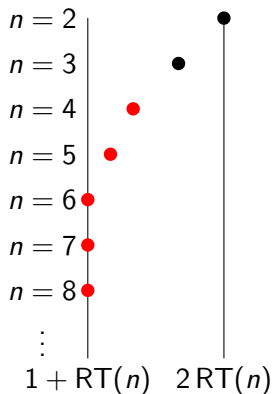
Repetition threshold for circular factors

Definition

The *repetition threshold for circular factors*, $RTC(n)$, is the smallest α for which there exists an infinite circularly α^+ -power-free word over Σ_n .

n	$RT(n)$	$RTC(n)$
2	2	4
3	$\frac{7}{4}$	$\frac{13}{4}$
4	$\frac{7}{5}$	$\frac{5}{2}$
5	$\frac{5}{4}$	$\frac{105}{46}$
6	$\frac{6}{5}$	$1 + \frac{6}{5} = \frac{11}{6}$
\vdots	\vdots	\vdots
k	$\frac{k}{k-1}$	$1 + RT(k) = \frac{2k-1}{k-1}$

Bounds on $RTC(n)$



Theorem

$$1 + RT(n) \leq RTC(n) \leq 2RT(n)$$

Thue-Morse word and RTC(2)

- Thue morphism

$$h(0) = 01$$

$$h(1) = 10.$$

- The *Thue-Morse word*

$$\mathbf{t} = h^\omega(0) = 01101001 \dots$$

is 2^+ -power-free.

Theorem

\mathbf{t} is circularly 4^+ -power-free.

Thue-Morse word and RTC(2)

Theorem

\mathbf{t} is circularly 4^+ -power-free.

Proof.

- Suppose (x, y) is a circular 4^+ -power of \mathbf{t} , i.e.,

$$\mathbf{t} = \boxed{\quad} \boxed{x} \boxed{\quad} \boxed{y} \boxed{\quad} \dots$$

and yx is a 4^+ -power.

- Then either y or x is a 2^+ -power, a contradiction.



$$\text{RTC}(2) = 4$$

Theorem

$$\text{RTC}(2) = 4.$$

Proof.

- Since \mathbf{t} is circularly 4^+ -power-free, we have

$$\text{RTC}(2) \leq 4.$$

- No binary word of length 12 is circularly 4-power-free, so

$$\text{RTC}(2) \geq 4.$$



$$\text{RTC}(3) = \frac{13}{4}$$

Overview of the proof:

- A finite search shows that $\text{RTC}(3) \geq \frac{13}{4}$.
- So to prove $\text{RTC}(3) = \frac{13}{4}$, we just need to construct an infinite word that is circularly $(\frac{13}{4})^+$ -power-free.
- We give a pair of morphisms:

$$\psi : \Sigma_6^* \rightarrow \Sigma_6^*$$

$$\mu : \Sigma_6^* \rightarrow \Sigma_3^*$$

- We prove $\mu(\psi^\omega(0))$ is circularly $(\frac{13}{4})^+$ -power-free.

Pair of morphisms

$$\psi(0) = 0435$$

$$\psi(1) = 2341$$

$$\psi(2) = 3542$$

$$\psi(3) = 3540$$

$$\psi(4) = 4134$$

$$\psi(5) = 4105.$$

$$\mu(0) = 012102120102012$$

$$\mu(1) = 201020121012021$$

$$\mu(2) = 012102010212010$$

$$\mu(3) = 201210212021012$$

$$\mu(4) = 102120121012021$$

$$\mu(5) = 102010212021012.$$

$$\mu(\psi^\omega(0))$$

Theorem

$\mu(\psi^\omega(0))$ is circularly $(\frac{13}{4})^+$ -power-free.

Proof idea

The proof has two parts

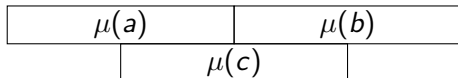
- 1 $\mathbf{r} = \psi^\omega(0)$ is circularly cubefree.
- 2 $\mathbf{s} = \mu(\mathbf{r})$ is circularly $(\frac{13}{4})^+$ -power-free.
 - 1 \mathbf{s} has no short circular $(\frac{13}{4})^+$ -power. (This is checked by computer)
 - 2 \mathbf{s} has no long circular $(\frac{13}{4})^+$ -power.

μ is well-behaved!

- $\mu : \Sigma_6^* \rightarrow \Sigma_3^*$ is 15-*uniform*

$$|\mu(a)| = 15 \text{ for all } a \in \Sigma_6.$$

- μ is *synchronizing*, i.e., for no $a, b, c \in \Sigma_6$



- μ is *strongly synchronizing*, i.e., for all $a, b, c \in \Sigma_6$ and $x, y \in \Sigma_3^*$ if

$$\mu(a) = \begin{array}{|c|c|} \hline x & \\ \hline \end{array}$$

$$\mu(b) = \begin{array}{|c|c|} \hline & y \\ \hline \end{array}$$

$$\mu(c) = \begin{array}{|c|c|} \hline x & y \\ \hline \end{array}$$

either $c = a$ or $c = b$.

Main lemma

Lemma

- Let ϕ be a strongly synchronizing q -uniform morphism.
- Let w be a circularly cubefree word.
- If (x_1, x_2) is a circular $(\frac{13}{4})^+$ -power in $\phi(w)$, i.e.,

$$\phi(w) = \boxed{\quad} \boxed{x_1} \boxed{\quad} \boxed{x_2} \boxed{\quad}$$

x_2x_1 is $(\frac{13}{4})^+$ -power, then

$$|x_2x_1| < 22q.$$

Main lemma: proof

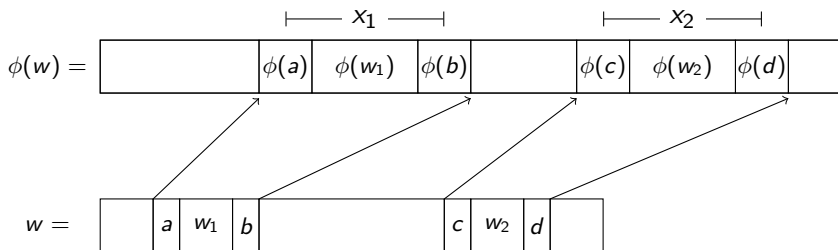
- Proof is by contradiction.
- Suppose (x_1, x_2) is a circular $(\frac{13}{4})^+$ -power of $\phi(w)$, and $|x_2 x_1| \geq 22q$.

$$\phi(w) = \boxed{\hspace{15cm}}$$

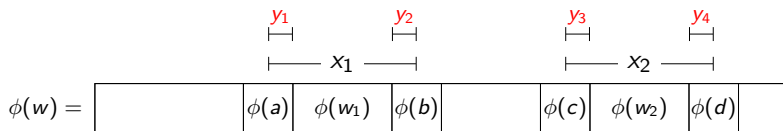
|----- x_1 -----| |----- x_2 -----|

Main lemma: proof

- Proof is by contradiction.
- Suppose (x_1, x_2) is a circular $(\frac{13}{4})^+$ -power of $\phi(w)$, and $|x_2 x_1| \geq 22q$.



Main lemma: proof



$$x_1 = \begin{array}{|c|c|c|} \hline y_1 & \phi(w_1) & y_2 \\ \hline \end{array}$$

$$x_2 = \begin{array}{|c|c|c|} \hline y_3 & \phi(w_2) & y_4 \\ \hline \end{array}$$

$$z^\alpha = x_2 x_1 = \begin{array}{|c|c|c|c|c|c|} \hline y_3 & \phi(w_2) & y_4 & y_1 & \phi(w_1) & y_2 \\ \hline \end{array}$$

Here z is a word and $\alpha > \frac{13}{4}$. There are two cases to consider:

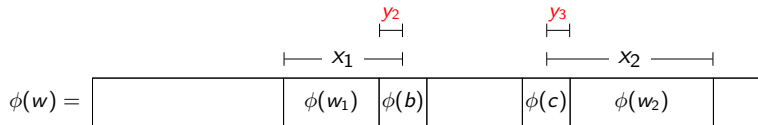
- ① $y_4 y_1 = \epsilon$
- ② $y_4 y_1 \neq \epsilon$

Main lemma: case 1

Case 1 is $y_4y_1 = \epsilon$. Therefore we have

$$x_1 = \phi(w_1)y_2,$$

$$x_2 = y_3\phi(w_2).$$



$$z^\alpha = x_2x_1 = \begin{array}{|c|c|c|c|} \hline y_3 & \phi(w_2) & \phi(w_1) & y_2 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline y_3 & \phi(w_2w_1) & y_2 \\ \hline \end{array}$$

- Note that $\alpha > \frac{13}{4} = 3.25$.
- We get that $\phi(w_2w_1)$ contains a cube.
- w_2w_1 contains a cube. (synchronizing)
- w contains a circular cube, a contradiction.

Main lemma: case 2

$$\phi(w) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & \phi(a) & \phi(w_1) & \phi(b) & & \phi(c) & \phi(w_2) & \phi(d) \\ \hline \end{array}$$

|----- x_1 -----| |----- x_2 -----|

- We would like to show that by shrinking x_1 and enlarging x_2 we can get another circular $(\frac{13}{4})^+$ -power of the same length:

$$\phi(w) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & \phi(a) & \phi(w_1) & \phi(b) & & \phi(c) & \phi(w_2) & \phi(d) \\ \hline \end{array}$$

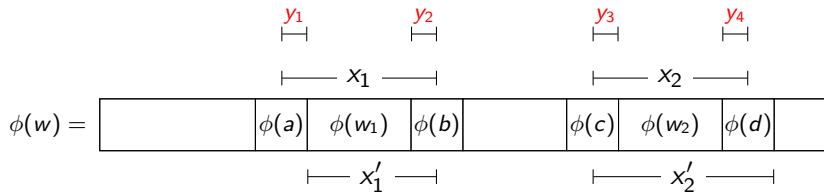
|----- x_1 -----| |----- x_2 -----|

|----- x'_1 -----| |----- x'_2 -----|

- Then clearly (x'_1, x'_2) falls under case 1.

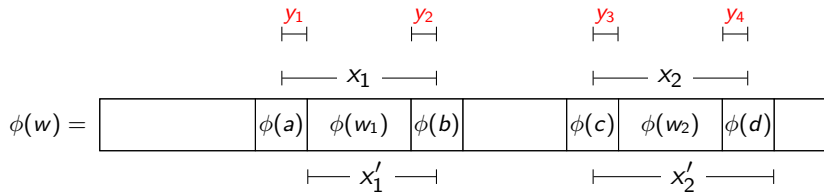
Main lemma: case 2

- Goal: $x_2'x_1' = x_2x_1$

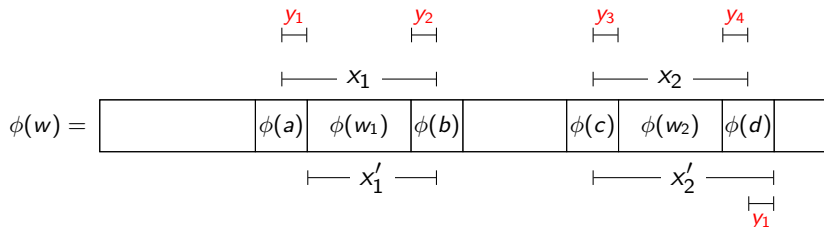


Main lemma: case 2

- Goal: $x'_2 x'_1 = x_2 x_1$

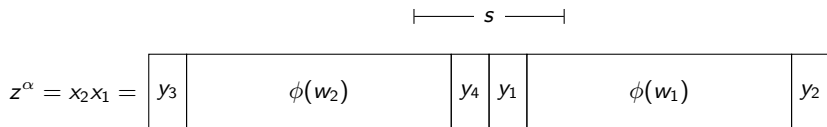


- We just need to show that $\phi(d) = y_4 y_1$.

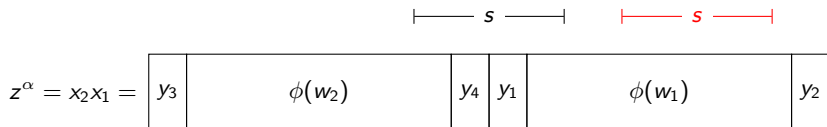


Main lemma: case 2

- Let $s = \phi(e)y_4y_1\phi(f)$, where e is the last letter of w_2 and f is the first letter of w_1 .



- s also appears in $\phi(w_1)$:



Main lemma: case 2

- Since ϕ is synchronizing, the word y_4y_1 is the complete image of a letter:

$$\phi(w) = \begin{array}{|c|c|c|c|c|c|} \hline & & s = & \phi(e) & y_4y_1 & \phi(f) \\ \hline \phi(w[0]) & \cdots & \phi(w[i]) & \phi(w[i+1]) & \phi(w[i+2]) & \cdots \\ \hline \end{array}$$

- Recall that y_4 is a prefix of $\phi(d)$ and y_1 is suffix of $\phi(a)$.

$$\phi(d) = \begin{array}{|c|c|} \hline y_4 & \\ \hline \end{array}$$

$$\phi(a) = \begin{array}{|c|c|} \hline & y_1 \\ \hline \end{array}$$

$$\phi(w[i+1]) = \begin{array}{|c|c|} \hline y_4 & y_1 \\ \hline \end{array}$$

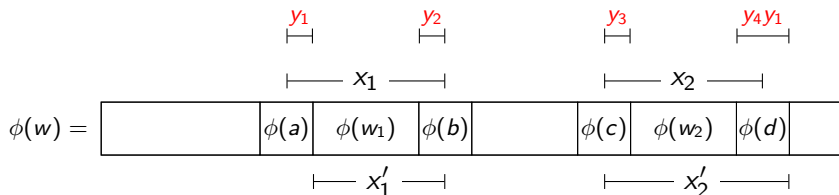
- Since ϕ is strongly synchronizing, we have either

$$y_4y_1 = \phi(d) \text{ or}$$

$$y_4y_1 = \phi(a).$$

Main lemma: case 2

- Without loss of generality, we can assume $y_4y_1 = \phi(d)$.



- So $x'_2x'_1 = x_2x_1$.
- (x'_1, x'_2) falls under case 1.
- A contradiction.

Open problem 1

- Prove or disprove

$$\text{RTC}(4) = \frac{5}{2},$$

$$\text{RTC}(5) = \frac{105}{46}, \text{ and}$$

$$\text{RTC}(n) = 1 + \text{RT}(n) = \frac{2n-1}{n-1} \text{ for } n \geq 6.$$

Open problem 2: generalized circular repetitions

- We can study repetition avoidance in the products of factors of words.
- Let RT_k denote the repetition threshold for this new problem, where k is the number of factors we take into consideration.

- We can easily prove

$$RT_k(2) = 2k.$$

- It would be interesting to obtain more values of $RT_k(n)$.
- Conjecture:

$$RT_2(n) = RTC(n)$$

- For large integers n , we conjecture that

$$RT_k(n) = k - 1 + RT(n).$$

Open problem 3: algorithmic problems

- For a finite word w , define the circular exponent, $\text{cexp}(w)$, to be

$$\text{cexp}(w) = \max\{\alpha : w \text{ has a circular } \alpha\text{-power}\}.$$

Is $\text{cexp}(w)$ computable in linear time?

- Given α and w , can we compute in linear time whether w avoids circular α -powers?