Some results on inhomogeneous percolation

E.J. Janse van Rensburg
Mathematics and Statistics, York University

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Collaborators: Neal Madras and Gary Iliev
### Inhomogeneous percolation

#### Introductory remarks

- Pictures

### A model of inhomogeneous percolation

<table>
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<th>$d$ dimensional</th>
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$s$ dimensional density $= \sigma$

dimensional density $= p$
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dimensional density = $p$

s dimensional density = $\sigma$

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Some results on inhomogeneous percolation
Inhomogeneous percolation: $30^2 \times 15, p = 0.2, \sigma = 0.4$
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Set up different densities in $\mathbb{L}_0$ and $\mathbb{L} \setminus \mathbb{L}_0$: 
Define

$$P^I_{p,\sigma}(e \text{ is open}) = \begin{cases} p, & \text{if } e \in \mathbb{L} \setminus \mathbb{L}_0; \\ \sigma, & \text{if } e \in \mathbb{L}_0. \end{cases}$$
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$\sigma = p \rightarrow$ homogeneous percolation: $P^I_{p,p} \equiv P^H_p$.

$$P^H_p(e \text{ is open}) = p, \quad \text{for } e \in \mathbb{L}.$$
If $\sigma = p$ then ordinary percolation is recovered:

$$\theta^H(p) \equiv \theta^I(p, p).$$
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There is a critical point $p_c$ such that

$$\theta^H(p) \begin{cases} 
= 0, & \text{if } p < p_c; \\
> 0, & \text{if } p > p_c.
\end{cases}$$
If $\sigma = p$ then ordinary percolation is recovered:

$$\theta^H(p) \equiv \theta^I(p, p).$$

There is a critical point $p_c$ such that

$$\theta^H(p) \begin{cases} = 0, & \text{subcritical phase;} \\ > 0, & \text{supercritical phase;} \end{cases}$$
The probability that the cluster at the origin is large

\[ P_{p,\sigma}(|C| \geq n) \]
The probability that the cluster at the origin is infinite

$$\lim_{n \to \infty} P_{p, \sigma}(|C| \geq n)$$
The probability that the cluster at the origin is infinite

\[ \theta^I(p, \sigma) = \lim_{n \to \infty} P^I_{p, \sigma}(|C| \geq n) = P^I_{p, \sigma}(|C| = \infty). \]
The probability that the cluster at the origin is infinite

$$\theta^I(p, \sigma) = \lim_{n \to \infty} P_{p,\sigma}^I(|C| \geq n) = P_{p,\sigma}^I(|C| = \infty).$$

Percolation:

$$\theta^I(p, \sigma) \begin{cases} = 0, & \text{if } p \text{ and } \sigma \text{ are small;} \\ > 0, & \text{if } p \text{ or } \sigma \text{ are large.} \end{cases}$$
The probability that the cluster at the origin is infinite

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Percolation:

\[ \theta^I(p, \sigma) \begin{cases} = 0, & \text{subcritical phase } \mathcal{R}_0; \\ > 0, & \text{supercritical phase.} \end{cases} \]

There is an infinite cluster at the origin with positive probability in the supercritical phase.
$2 \leq s < d$: $\mathbb{L}$ is $d$-dimensional, $\mathbb{L}_0$ is $s$-dimensional.
2 ≤ s < d: \( \mathbb{L} \) is \( d \)-dimensional, \( \mathbb{L}_0 \) is \( s \)-dimensional.

If \( p = 0 \): The probability of percolation in \( \mathbb{L}_0 \) is

\[
\theta'(0, \sigma)
\]
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\theta^I(0, \sigma) = \theta^H_s(\sigma) = \lim_{n \to \infty} P^H_\sigma(|C| \geq n) = P^H_\sigma(|C| = \infty).
\]

with critical density \( \sigma_c = p_c(s) \).
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with critical density \( \sigma_c = p_c(s) \).

\( \theta^I(p, \sigma) \) is non-decreasing with \( p \) and \( \sigma \):

\[
\theta^H_s(\sigma) = \theta^I(0, \sigma) \leq \theta^I(p, \sigma).
\]

That is, if \( \sigma > \sigma_c = p_c(s) \), then \( \theta^I(p, \sigma) > 0 \).
Surface transition:

If \( s \geq 2 \) then \( 0 < \sigma_c = \sigma^*(0) < 1 \).
The critical curve $\sigma^*(p)$

Surface transition:

If $s \geq 2$ then $0 < \sigma_c = \sigma^*(0) < 1$.

Define the critical surface density $\sigma^*(p)$ for all $p \in [0, 1]$:

$$\sigma^*(p) = \inf\{\sigma \mid \theta(p, \sigma) > 0\}$$
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If $p < p_c(d)$ and $\sigma = \sigma^*(p)$: Surface transition
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$\theta(p, \sigma)$ is non-decreasing $\Rightarrow \sigma^*(p)$ is non-increasing with $p$. 

$0 < \sigma^*_c(0) = p_c(s) < 1$, $\Rightarrow \sigma^*(p) < 1 \forall p \in [0, 1]$. 
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The critical curve $\sigma^*(p)$

Bulk transition:
Percolation in a half-space
Bulk transition:
Percolation in a half-space – percolation probability $\theta^+(p)$

$$p \theta^+(p) \leq \theta^l(p, 0)$$
The critical curve $\sigma^*(p)$

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Bulk transition:
Percolation in a half-space – percolation probability $\theta^+(p)$

$$p \theta^+(p) \leq \theta^l(p,0) \leq \theta^l(p,\sigma)$$

If $p > p^+_c(d) = p_c(d)$ then $\theta^l(p,\sigma) > 0$ – Regime $\mathcal{R}_H$

$\sigma^*(p) = 0$ if $p > p_c(d)$.  

Bulk transition:
Percolation in a half-space – percolation probability $\theta^+(p)$

\[ p \theta^+(p) \leq \theta^I(p, 0) \leq \theta^I(p, \sigma) \]

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If $0 \leq \sigma < p < p_c(d)$,
Bulk transition:
Percolation in a half-space – percolation probability $\theta^+(p)$

$$p \theta^+(p) \leq \theta^I(p, 0) \leq \theta^I(p, \sigma) \leq \theta^I(p, p) = \theta^H(p).$$

If $p > p_c^+(d) = p_c(d)$ then $\theta^I(p, \sigma) > 0$ – Regime $R_H$

$\sigma^*(p) = 0$ if $p > p_c(d)$.

If $0 \leq \sigma < p < p_c(d)$,
**The critical curve** $\sigma^*(p)$

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$\sigma^*(p) = 0$ if $p > p_{c}(d)$.

If $0 \leq \sigma < p < p_{c}(d)$, then $\theta^I(p, \sigma) = 0$ – Regime $\mathcal{R}_0$

This shows that $\sigma^*(p) \geq p_{c}(d) > 0$ if $p < p_{c}(d)$
Inhomogeneous percolation

The critical curve and phase diagram

The phase diagram

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\[ \sigma^*(p) \]

\[ \theta^I(p, \sigma) = 0 \]

\[ \theta^I(p, \sigma) > 0 \]

\[ R_L \]

\[ R_H \]

\[ R_0 \]

[Graph showing the phase diagram with critical points and regions labeled.]
The phase diagram

\[ \theta^I(p, \sigma) > 0 \]

\[ \mathcal{R}_L \]

\[ \mathcal{R}_H \]

\[ \sigma^*(p) \]

\[ \mathcal{R}_0 \]

\[ \theta^I(p, \sigma) = 0 \]

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Some results on inhomogeneous percolation
Newman and Wu: $0 \ll s + 3 \leq d$:

\[
\begin{align*}
\sigma^*(p) & > 0 \\
\mathcal{R}_L & > 0 \\
\mathcal{R}_H & > 0
\end{align*}
\]
The phase diagram

Newman and Wu: $0 \ll s + 3 \leq d$:

- $\sigma^*(p)$
- $\theta^I(p, \sigma) = 0$ at $p_c(d)$
- $\theta^I(p, \sigma) > 0$ in $\mathcal{R}_H$
- $\theta^I(p, \sigma) > 0$ in $\mathcal{R}_L$

$\mathcal{R}_0$, $\mathcal{R}_L$, $\mathcal{R}_H$
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**The phase diagram**

Newman and Wu: \( 0 \ll s + 3 \leq d \):

\[
\theta^I(p, \sigma) > 0 \\
\mathcal{R}_L \\
\mathcal{R}_0 \\
\mathcal{R}_H \\
\theta^I(p, \sigma) = 0
\]
Newman and Wu: \(0 \ll s + 3 \leq d:\)

\[
\begin{align*}
\sigma^*(p) &< 0.25 \\
\theta^I(p, \sigma) &< 0 \\
\sigma &< 0.75 \\
\sigma &< 1
\end{align*}
\]

\[
\begin{align*}
\mathcal{R}_L &< \mathcal{R}_0 \\
\mathcal{R}_0 &< \mathcal{R}_H
\end{align*}
\]
Newman and Wu: $0 \ll s + 3 \leq d$:

\[ \theta^I(p, \sigma) > 0 \]
\[ \mathcal{R}_H \]
\[ \mathcal{R}_L \]
\[ \sigma^*(p) \]
\[ \mathcal{R}_0 \]

\[ \theta^I(p, \sigma) = 0 \]

\[ p_c(d) \]
The phase diagram

Newman and Wu: $0 \ll s + 3 \leq d$:

\[ \theta^I(p, \sigma) > 0 \quad R_H \]
\[ \theta^I(p, \sigma) = 0 \quad R_0 \]
\[ \theta^I(p, \sigma) < 0 \quad R_L \]
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Some results on inhomogeneous percolation
\[ \mathbb{L} = L^d \text{ cube with } \mathbb{L}_s = L^s \text{ defect plane.} \]
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Cluster connecting opposite sides of \( \mathbb{L} \? - \text{crossing probability.} \]
Wrapping probability

\( \mathbb{L} = L^d \) cube with \( \mathbb{L}_s = L^s \) defect plane.
Cluster connecting opposite sides of \( \mathbb{L} \)? – crossing probability.

Edges realised at density \( p \) in \( \mathbb{L} \setminus \mathbb{L}_s \), denoted by \( R \);
With \( t \) edges open in \( \mathbb{L}_s \).
Crossing probability = \( Q_{R,L}(t) \).
Wrapping probability

\( \mathbb{L} = L^d \) cube with \( \mathbb{L}_s = L^s \) defect plane.
Cluster connecting opposite sides of \( \mathbb{L} \) – crossing probability.

Edges realised at density \( p \) in \( \mathbb{L} \setminus \mathbb{L}_s \), denoted by \( R \);
With \( t \) edges open in \( \mathbb{L}_s \).
Crossing probability = \( Q_{R,L}(t) \).

\[
Q_{R,L}(\sigma) = \sum_{t=0}^{\lvert \mathbb{L}_s \rvert} \binom{\lvert \mathbb{L}_s \rvert}{t} \sigma^t (1 - \sigma)^{\lvert \mathbb{L}_s \rvert - t} Q_{R,L}(t).
\]
Wrapping probability

$L = L^d$ cube with $L_s = L^s$ defect plane.
Cluster connecting opposite sides of $L$? – crossing probability.

Edges realised at density $p$ in $L \setminus L_s$, denoted by $R$;
With $t$ edges open in $L_s$.
Crossing probability $= Q_{R,L}(t)$.

\[
Q_L(p, \sigma) = \langle Q_{R,L}(\sigma) \rangle_p = \sum_{t=0}^{\|L_s\|} \binom{\|L_s\|}{t} \sigma^t (1 - \sigma)^{\|L_s\| - t} \langle Q_{R,L}(t) \rangle_p.
\]
\( \mathbb{I} = L^d \) cube with \( \mathbb{I}_s = L^s \) defect plane.
Cluster connecting opposite sides of \( \mathbb{I} \)? – crossing probability.

Edges realised at density \( p \) in \( \mathbb{I} \setminus \mathbb{I}_s \), denoted by \( R \);
With \( t \) edges open in \( \mathbb{I}_s \).
Crossing probability = \( Q_{R,L}(t) \).

\[
Q_L(p, \sigma) = \langle Q_{R,L}(\sigma) \rangle_p = \sum_{t=0}^{\left| \mathbb{I}_s \right|} \binom{\left| \mathbb{I}_s \right|}{t} \sigma^t (1 - \sigma)^{\left| \mathbb{I}_s \right| - t} \langle Q_{R,L}(t) \rangle_p.
\]

One expects that:

\[
Q_L(p, \sigma) \rightarrow \begin{cases} 0, & \text{in } \mathcal{R}_0; \\ 1, & \text{in } \mathcal{R}_L \cup \mathcal{R}_H, \end{cases} \quad \text{as } L \to \infty.
\]
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Wrapping probability

Figure: $Q_L(0.1, \sigma)$ for $d = 3$ and $s = 2$
Wrapping probability

Figure: $Q_L(p_c(3), \sigma)$ for $d = 3$ and $s = 2$
Figure: $Q_L(p_c(4), \sigma)$ for $d = 4$ and $s = 2$
Decay of the cluster at the origin: Some theorems

**Theorem**

Suppose that \((p, \sigma) \in \mathcal{R}_0\). Then

\[
P_{p,\sigma}(|C| \geq n) \leq 2e^{-n/(2\chi_H(p)\chi_I(p,\sigma))}.
\]
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Decay of the cluster at the origin: Some theorems

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P_{p,\sigma}(|C| \geq n) \leq 2e^{-n/(2 \chi^H(p) \chi^I(p,\sigma))}.
\]

In particular

\[
\zeta^I(p, \sigma) = -\limsup_{n \to \infty} \frac{1}{n} \log P_{p,\sigma}(|C| = n) \\
\geq -\limsup_{n \to \infty} \frac{1}{n} \log P_{p,\sigma}(|C| \geq n) \geq \frac{1}{2 \chi^H(p) \chi^I(p,\sigma)}.
\]
Decay of the cluster at the origin: Some theorems

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\[
\geq -\limsup_{n \to \infty} \frac{1}{n} \log P_{p,\sigma}(|C| \geq n) \geq \frac{1}{2\chi^H(p)\chi^I(p,\sigma)} > 0.
\]

If \((p, \sigma) \in \mathcal{R}_0\).
Decay of the cluster at the origin: Some theorems

**Theorem (Aizenman, Delyon, Souillard 1980)**

If \((p, \sigma) \in \mathcal{R}_H\) there exists a \(\gamma(p) > 0\) such that

\[
P^I_{p,\sigma}(|C| \geq n) \geq e^{-\gamma(p) n^{(d-1)/d}}.
\]
Decay of the cluster at the origin: Some theorems

**Theorem (Aizenman, Delyon, Souillard 1980)**

If \((p, \sigma) \in \mathcal{R}_H\) there exists a \(\gamma(p) > 0\) such that

\[ P^I_{p, \sigma}(|C| \geq n) \geq e^{-\gamma(p) n^{(d-1)/d}}. \]

**Theorem**

If \((p, \sigma) \in \mathcal{R}_L\) then there exist positive \(\beta_1\) and \(\beta_2\)

\[ P^I_{p, \sigma}(|C| \geq n) \geq \beta_1 e^{- \left( \beta_2 n^{(s-1)/s} \left( \log^2 n \right)^{d-s} \right)} \]

The proof is adapted from the proof for homogeneous percolation by Aizenman, Delyon and Souillard (1980).
The function $\zeta^I(p, \sigma)$

$$\zeta^I(p, \sigma) = -\limsup_{n \to \infty} \frac{1}{n} \log P_{p,\sigma}(|C| = n)$$
The function $\zeta^I(p, \sigma)$

$$
\zeta^I(p, \sigma) = - \limsup_{n \to \infty} \frac{1}{n} \log P_{p, \sigma}(|C| = n) > 0
$$

if $(p, \sigma) \in \mathcal{R}_0$. 
The function $\zeta^l(p, \sigma)$

$$\zeta^l(p, \sigma) = -\limsup_{n \to \infty} \frac{1}{n} \log P_{p,\sigma}(|C| = n) > 0$$

if $(p, \sigma) \in R_0$.

Define

$|C| = \text{number of vertices in } C$;

$\|C\| = \text{number of edges in } C$.

Then $|C| - 1 \leq \|C\| \leq d |C|$.
The function $\zeta^I(p, \sigma)$

$$\zeta^I(p, \sigma) = - \limsup_{n \to \infty} \frac{1}{n} \log P_{p,\sigma}(|C| = n) > 0$$

if $(p, \sigma) \in R_0$.

Define

$|C| = \text{number of vertices in } C$;

$\|C\| = \text{number of edges in } C$.

Then $|C| - 1 \leq \|C\| \leq d |C|$.

Existence of the limits

$$\zeta^I(p, \sigma) = - \lim_{n \to \infty} \frac{1}{n} \log P^I_{p,\sigma}(|C| = n)$$

$$\psi^I(p, \sigma) = - \lim_{n \to \infty} \frac{1}{n} \log P^I_{p,\sigma}(\|C\| = n)$$

follows from a concatenation of clusters argument.
Inhomogeneous percolation
The critical curve and phase diagram

Lattice animals

\[ a_{n,m}(t,r) = \#\{\text{Lattice animals}\} \]

- at the origin
- size \(n\) edges
- \(m\) edges in \(E_0\)
- \(t\) perimeter edges in \(L \setminus L_0\)
- \(r\) perimeter edges in \(L_0\)

Partition function:

\[ Z^I_n(x,y,z) = \sum_{t \geq 0} \sum_{r \geq 0} \sum_{m \geq 0} a_{n,m}(t,r) x^t y^r z^m. \]

The limiting free energy is defined by

\[ F^I(x,y) = \lim_{n \to \infty} \frac{1}{n} \log Z^I_n(x,y,z). \]
Existence of the Free Energy

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**Existence of the Free Energy**

\[
Z_N(x, y, z) Z_M(x, y, z) \leq (N + M + 3)^3 \lambda(x, y, z) Z_{N+M+2}(x, y, z).
\]

\[
(Z_{N-2}(x, y, z)) / \lambda(x, y, z)
\]
satisfies a supermultiplicative inequality.

**Theorem**

For \(x, y, z \in (0, \infty)\) the limit

\[
\mathcal{F}^I(x, y, z) = \lim_{n \to \infty} \frac{1}{n} \log Z_n^I(x, y, z)
\]

exists. \(\mathcal{F}^I(x, y, z)\) is log-convex in each of its arguments.
One may show that (put $x = q = 1 - p$ and $y = \tau = 1 - \sigma$)

$$\mathcal{F}^l(q, \tau, \sigma/p) = -\log p - \psi^l(p, \sigma)$$
The function $\zeta^I(p, \sigma)$

One may show that (put $x = q = 1 - p$ and $y = \tau = 1 - \sigma$)

$$F^I(q, \tau, \sigma / p) = -\log p - \psi^I(p, \sigma)$$

\[
\begin{cases}
  = -\log p, & \text{in } \mathcal{R}_H; \\
  < -\log p, & \text{in } \mathcal{R}_0.
\end{cases}
\]

This proves existence of $\psi^I(p, \sigma)$ and $\zeta^I(p, \sigma)$ as limits.
The function $\zeta^I(p, \sigma)\)\\\\One can show that $\zeta^I(p, \sigma) > 0$ if and only if $\psi^I(p, \sigma) > 0$.\)
The function $\zeta^I(p, \sigma)$

One can show that $\zeta^I(p, \sigma) > 0$ if and only if $\psi^I(p, \sigma) > 0$.

If $(p, \sigma) \in \mathcal{R}_0$, then

$$\zeta^I(p, \sigma) > 0 \quad \text{and} \quad \psi^I(p, \sigma) > 0$$

If $(p, \sigma) \in \mathcal{R}_H$ (for $p > p_c(d)$), then

$$\zeta^I(p, \sigma) = 0 \quad \text{and} \quad \psi^I(p, \sigma) = 0$$

If $(p, \sigma) \in \mathcal{R}_L$ (for $p < p_c(d)$ and $\sigma > \sigma^*(p)$), then

$$\zeta^I(p, \sigma) = 0 \quad \text{and} \quad \psi^I(p, \sigma) = 0$$
Inhomogeneous percolation

The critical curve and phase diagram

The phase diagram

The wrapping probability

Decay of the infinite cluster

The function $\zeta^I(p, \sigma)$

Concatenation

The surface transition and Conclusions

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The phase diagram

\[ \sigma^*(p) \]

\[ \zeta^I(p, \sigma) > 0 \]

\[ \zeta^I(p, \sigma) = 0 \]

\[ R_L \]

\[ R_0 \]

\[ R_H \]

Some results on inhomogeneous percolation

E.J. Janse van Rensburg, Mathematics and Statistics, York University
A bound on $\sigma^*(p)$

$s$ dimensional adsorbing plane

$\mathbb{L}_0$
A bound on $\sigma^*(p)$

$s$ dimensional adsorbing plane

$2s \leq d$
### A bound on $\sigma^*(p)$

A bound on $\sigma^*(p)$ for the $s$ dimensional adsorbing plane is given by:

\[ 2s \leq d \]

where $s$ is the dimension of the plane and $d$ is the dimension of the space. This inequality ensures that the system does not become too complex for the bound to hold.
A bound on $\sigma^*(p)$

$s$ dimensional adsorbing plane

$2s \leq d$
A bound on $\sigma^*(p)$

$s$ dimensional adsorbing plane

$2s \leq d$
A bound on $\sigma^*(p)$

$s$ dimensional adsorbing plane

Percolation in $\mathbb{L}_0$ at density $p_s(p, \sigma)$

$2s \leq d$

$\mathbb{L}_0$
A bound on $\sigma^*(p)$

$s$ dimensional adsorbing plane

Percolation in $\mathbb{L}_0$ at density $p_s(p, \sigma)$

Probabilities: $\theta^S(p, \sigma) \leq \theta^I(p, \sigma)$
A bound on $\sigma^*(p)$

$s$ dimensional adsorbing plane

Percolation in $\mathbb{L}_0$ at density $p_s(p, \sigma)$
Probabilities: $\theta^S(p, \sigma) \leq \theta^I(p, \sigma)$

If $p_s > p_c(s)$ then $\theta^S(p, \sigma) > 0$
A bound on $\sigma^*(p)$

Inhomogeneous percolation
The critical curve and phase diagram

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$\mathbb{L}_0$

$2s \leq d$

Percolation in $\mathbb{L}_0$ at density $p_s(p, \sigma)$

Probabilities: $\theta^S(p, \sigma) \leq \theta^I(p, \sigma)$

If $p_s > p_c(s)$ then $\theta^S(p, \sigma) > 0$

But $p_s \geq (1 - \sigma)(1 - p^3)$

Solve to get a bound on $\sigma^*(p)$
A bound on $\sigma^*(p)$

**Theorem**

Suppose that $p \in (0, p_c(d))$ and $d \geq 2s$. Then the critical curve $\sigma^*(p)$ is bounded above by

$$\sigma^*(p) \leq 1 - \frac{1 - p_c(s)}{1 - p^3} = p_c(s) - (1 - p_c(s))p^3 + O(p^6).$$

The RHS is a decreasing function of $p$, hence

$$p_c(s) = \sigma^*(0) > \sigma^*(p), \quad \text{for } p > 0.$$ 

One may show that $\sigma^*(p)$ is strictly decreasing for $p \in [0, p_c(d))$. 

E.J. Janse van Rensburg, Mathematics and Statistics, York University

Some results on inhomogeneous percolation
Some basic results are easily obtained from homogeneous percolation.

Other results require generalisation of differential inequalities, and bounds on percolation functions (e.g., susceptibility) in terms of homogeneous percolation functions.

Both the surface transition and bulk transition are percolation phenomena.

Are the surface phase and bulk phase different thermodynamic phases?

What is the nature of the transition if $p = p_c(d)$?