Ruskey’s Open Problem On Hofstadter’s Q Function

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Hostadter Q Function

- $Q(n) = Q(n-Q(n-1))+Q(n-Q(n-2))$
- Possible:
- To use subscript notation Q_n
- To use Vector Notation: $Q_{1:3} = <1,2,3>$
- To use range notation: $Q(1:3) = <1,2,3>$

Compact Vector Notation

- $Q_{1:oo} = <3, 3k-1, 3k-2>_{k \geq 1}$
- 1) Angle brackets indicate sequence vs. set
- 2) Subscripts indicate repeating seq. blocks
- $<3, 3k-1, 3k-2>_{k \geq 1} = 3,2,1 \quad 3,5,4 \quad 3,8,7 \ldots$
- 3) Compact theorem notation
- $Q_{1:3}=<3,2,1>$ ➔ $Q_{1:oo} = <3, 3k-1, 3k-2>_{k \geq 1}$
- Perhaps, drop $k \geq 1$ (like SAS vector notation)

More Vector Notation

- 4) Use of exponents to indicate repetition
- $<5,2>^3$ = <5,2   5,2   5,2>
- Can think of it as regular expressions w variables
- 5) Nesting, indicated by parenthesis, useful:
- $(\quad <6k,2>^{(6-2)/2} \; U \; < 6k, 4+2 > \quad)_{k \geq 1} \; =$
- <6,2,6,2   6,6   12,2,12,2   12,6   18,2,18,2, 18,6>
- 6) Use double vertical line to indicate if-then
- <3,2,1 || 3,5,4   3,8,7   3,11,10…

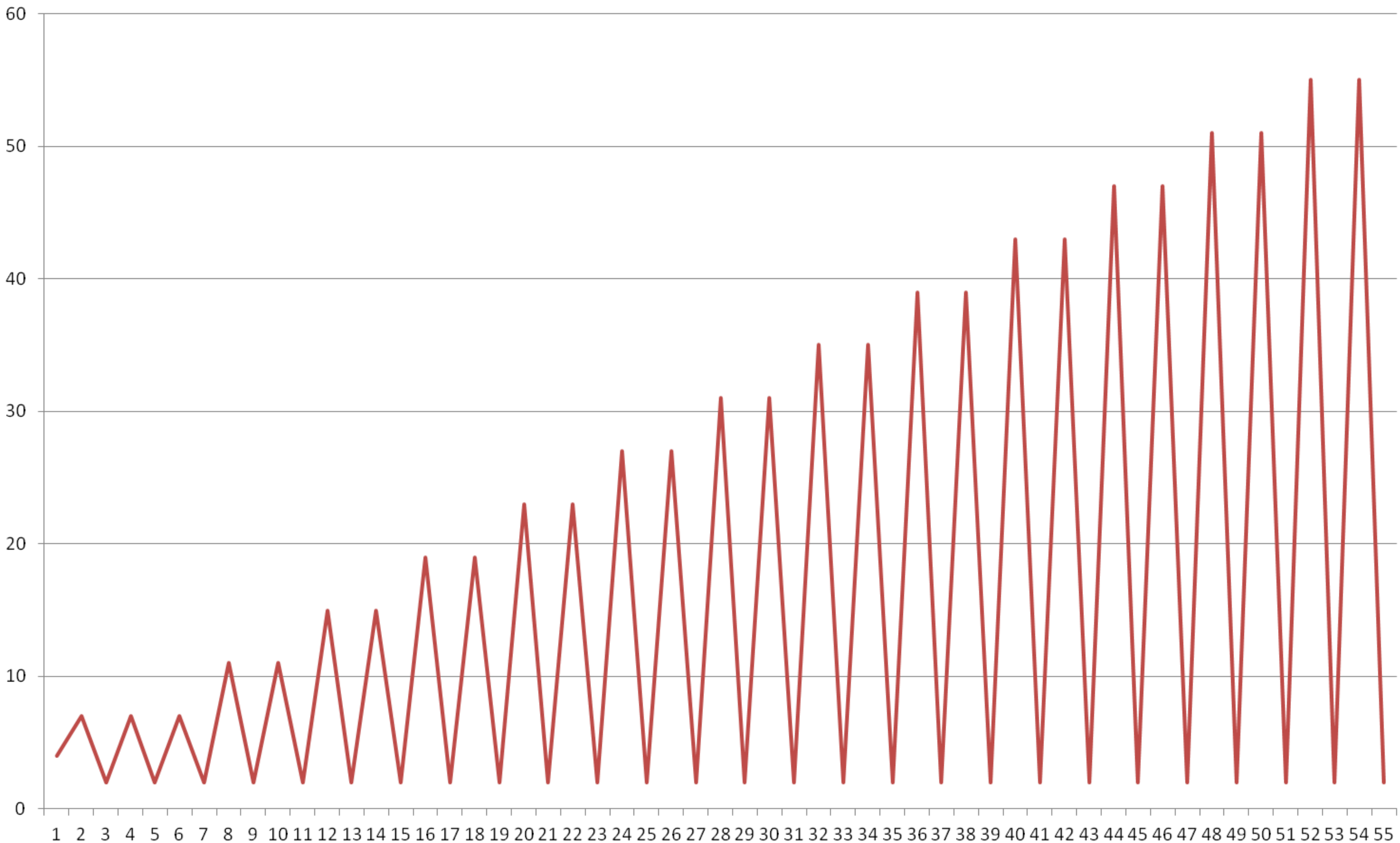
Ruskey's Open Problem Background

- <u>Golumb's</u> result/example of Hofstadter *Q:*
- $Q_{1:3} = <3,2,1> \Rightarrow Q_{1:oo} = <3,3k-1,3k-2>_{k \geq 1}$
- <u>Ruskey's</u> result/example of Hofstadter *Q:*
- $Q_{0:5} = <3,6,5,3,6,8> \rightarrow Q_{0:oo} = <3,6,F_{k+4}>_{k \geq 1}$
- --
- <u>Ruskey's open problem</u>: Do all nonterminating examples have Period 3?
- <u>Hendel's answer</u>: For every even *e*, there is a *Q* sequence with eventual exact period *e*?

Example

- Next slide presents nested recursion:
- Sequence with eventual period 4
- This example illustrates THEOREM a) in the second following slide

$$4\ (7,2)^3\ ||\ (11,2)^2\ (15,2)^2\ (19,2)^2$$



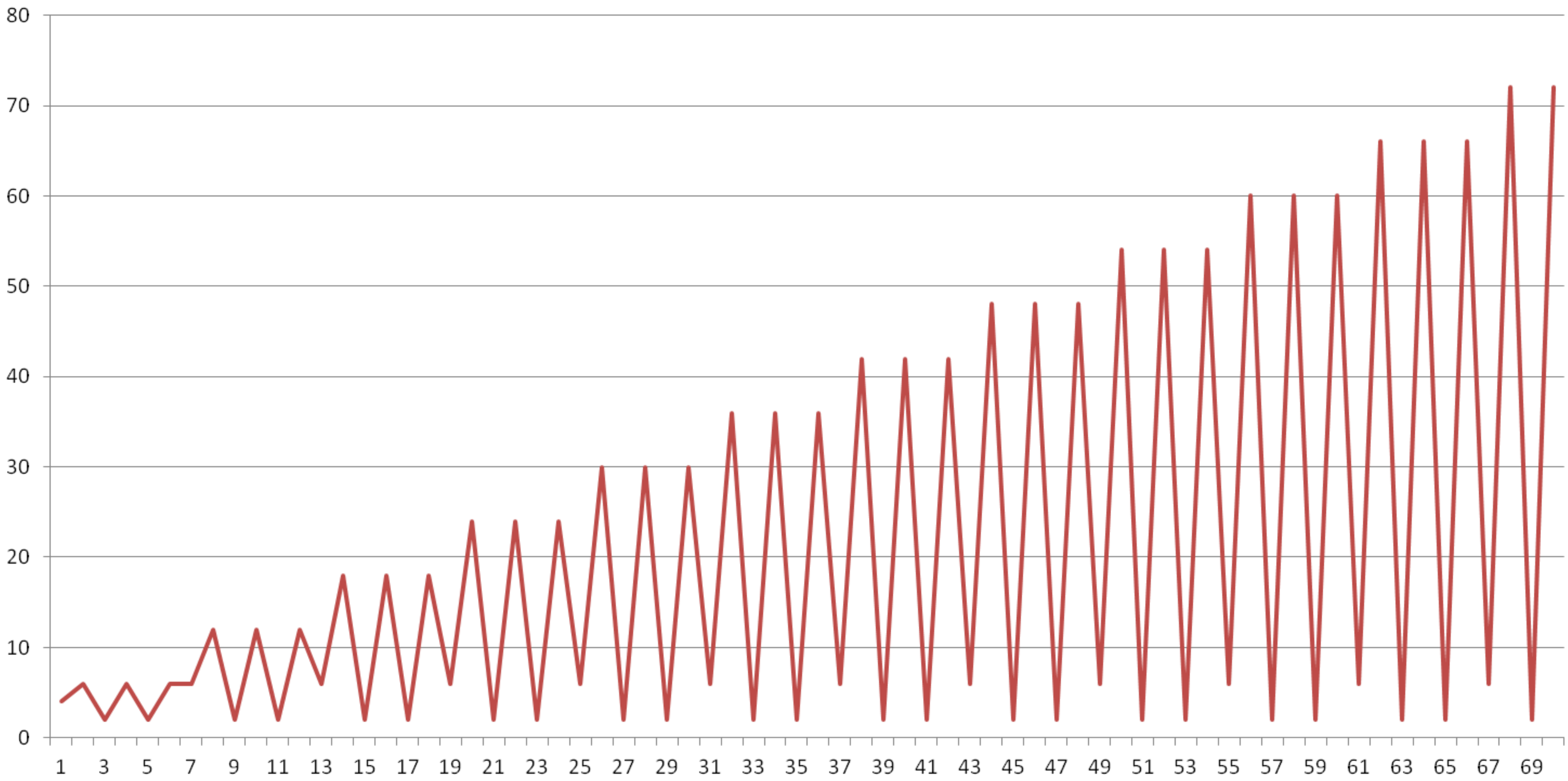
Theorem Formulation

- Let *o,e* indicate variables over odd/even #
- *THEOREM a)* $o > e \Rightarrow Q_{1:3} = <e,o,2> \Rightarrow$
- $Q_{1:\infty} = <e> \cup <o,2>^{(o-1)/2} \cup <o+ek,2>^{e/2}{}_{k \geq 1}$
- *THEOREM b)* $e_2 > e_1 \Rightarrow Q_{1:3} = <e_1,e_2,2> \Rightarrow$
- $Q_{1:\infty} = <e_1> \cup <ke_2,2>^{(e_2-2)/2} \cup <ke_2,e_1+2>_{k \geq 1}$
- --
- Corollary: For every *e*>0, there is a *Q* sequence with exact period *e*.

Example

- Next slide
- Presents example of THEOREM b)
- Sequence with eventual period 6
- Note the “subpatterns”

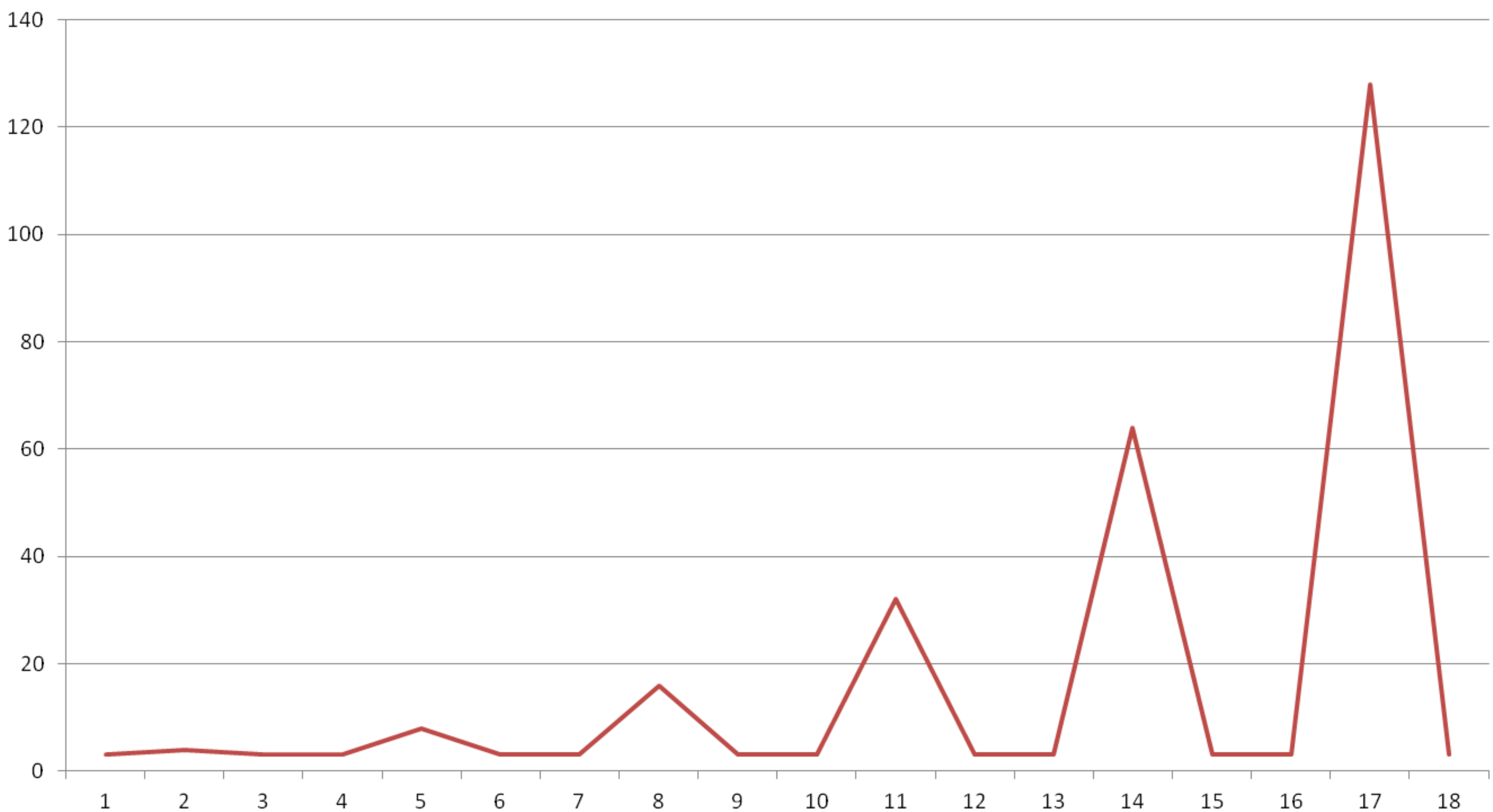
$4\ (6,2)^2\ 6,6\ \ ||\ (12,2)^2\ 12,6\ \ (18,2)^2\ 18,6$



Example Geometric

- Next slide
- Example of Geometric *Q* sequence
- Note alternation:
- constant,geometric,constant
- Differs from non-nested recursions

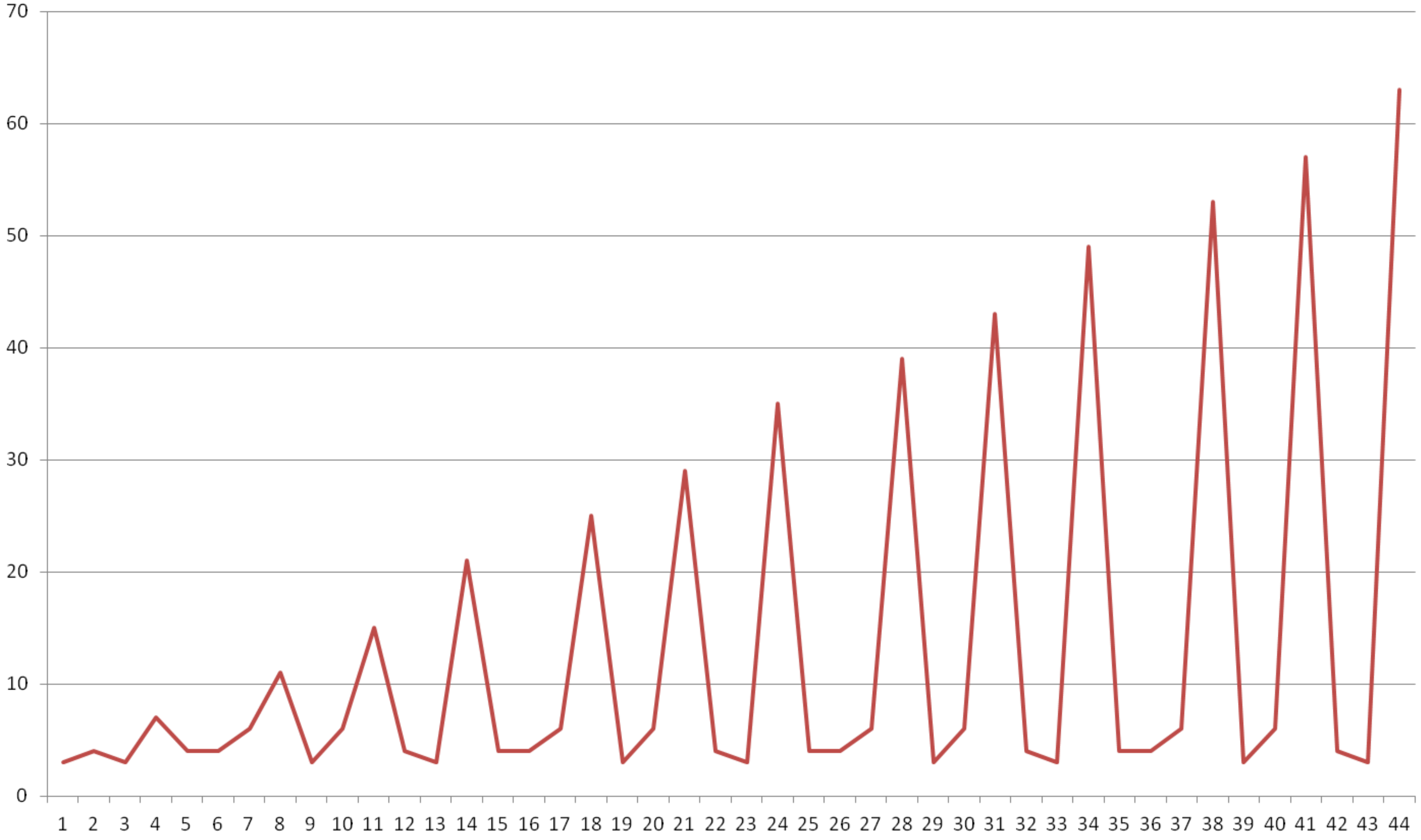
$$<3,2^{k+1},3>_{k\geq 1} = 3,4,3, \;||\; 3,8,3 \quad 3,16,3 \ldots$$



Example

- Next slide presents nested recursion, an
- Example with period 10
- Each period has 3 sub-periods
- Sub periods have constant and linear terms
- Illustrates the complexity of nested recursions

3,4,3,7 || 4,4,6,14k-3 3,6,14k+1 4,3,14k+7



A Complex Example

- $Q_{1:3} = <1,5,3>$
- Consequences
- $Q_{4:14} = <1,6,6,2,11,2,11,2,12,3,7>$, Random
- Now follows a period-4 set of segments
- Segment entries are geometric+periodic noise
- Segment lengths grow geometrically+noise

Complex Example continued

- $Q_{15:oo} = S_0\ S_1\ S_2\ S_3 \ldots$ with
- $S_i = \langle x_i, 2\rangle^{(L_i\ -\ 3)/2}\quad U \quad \langle y_{i,}\ 3,7\rangle$
- With L_0=11, $L_i = 2L_{i\ -1} + e_i$ with
- $e_{0:oo} = \langle 1, 1, -1, -1\rangle^{oo}$ and with
- x_0 = 11, $x_i = 2x_{i-1} + e'_i$ with
- $e'_{0:oo} = \langle 0,0,1,1\rangle^{oo}$ and with
- $y_i = x_i + e'_i$

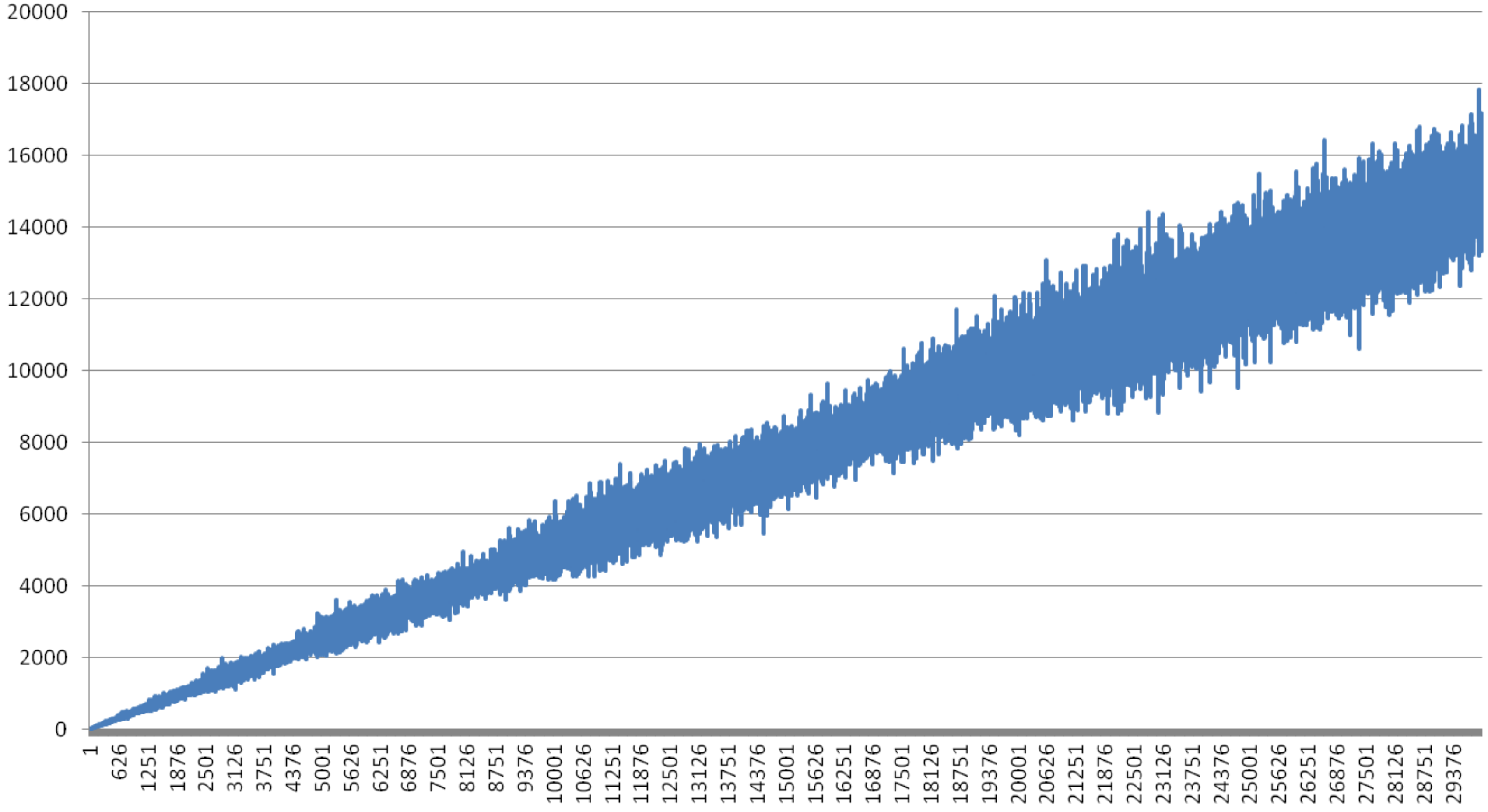
Complex Example Numerical Details

- $Q_{15:oo} = S_0\ S_1\ S_2\ S_3 \ldots$
- $S_0 = Q_{15:25} = <23,2>^4\ <23,3,7>$ Length=L_0 =11
- $S_1 = Q_{26:48} = <46,2>^{10}\ <46,3,7>$ Length L_1=23
- $S_2 = Q_{49:95} = <92,2>^{22}\ <93,3,7>$ Length L_2=47
- $S_3 = Q_{96:188} = <185,2>^{45}\ <186,3,7>$ L_3 =93

Final Example: Asymptotic

- Next slide presents following nested recursion
- Hofstadter Q function with
- Initial conditions $Q_{1:6}$ = <3,2,3,4,5,6>
- The sequence appears to be random but
- Has upper and lower bounds
- Perhaps these bounds are lines
- Perhaps they are polynomials or exponentials
- Several questions raised after graph

Asymptotic Example



Questions and comparisons

- Consider non-nested recursion case
- E.g. The Fibonacci numbers
- They have closed Binet form $F_n=(a^n-b^n)/\text{sqrt}(5)$
- *a,b* quadratic irrationals
- So Easy to get explicit asymptotic bounds
- --
- For nested recursion case, difficult
- Closed forms (so far) in only rare cases
- How do you do even simple estimates
- Also: There are multiple subsequences

Nested Recursions: Nested Proofs

- Proofs of results have unique features
- Typical <u>steps</u> are routine inductions
- But the <u>overall</u> proof structure is complex
- Overall proof structure = Tree structure
- Proofs themselves are nested

Statement of Theorem (a)

- Assume $Q_{1:3}$ = <*e*,*o*,2>, *o* > *e*, then
- $Q_{1:3}$=<*e*,*o*,2> & *o*>*e* ≥ 2 ➔
- $Q_{1:oo}= e \quad U \quad <o,2>^{(o-1)/2} \quad U \quad <o+ek,2>^{e/2}$
- ---
- Example given previously
- 4 7,2,7,2,7,2 11,2,11,2 15,2,15,2 ….

Proof: First level

- Assume
- Equation (0): $Q_{1:3} = \langle e,o,2\rangle$, $o > e \geq 2$.
- Then
- (1) Equation (0) ➔ $Q(2{:}o) = \langle o,2\rangle^{(o-1)/2}$
- (2) $Q(\mathrm{o+ne-1{:}\ o+ne}) = \langle o+ne,2\rangle$ ➔
- $Q(\mathrm{o+ne+1{:}\ o+(n+1)e}) = \langle o+(n+1)e,2\rangle^{e/2}$
- Eq. (1)(2) do prove THEOREM a).
- Problem: Eq (1) and (2) must now each be proved. “Nested” proofs

Proof of Eq. (1)

- Eq. (1) states as follows:
- Equation (0) ➔ $Q(2: o) = <o,2>^{(o-1)/2}$
- <u>Proof</u>
- Assume $o \geq 5$ and $o' \leq o-2$
- (1a) Show $Q(o'-1: o') = <o,2> =>$
- $Q(o'+1: o'+2) = <o,2>$
- Last statement uses vector notation
- Statement has two assertions
- Both are plug-ins from definition of Q

Proof of Equation (2)

- Statement of Eq. (2):
- $Q(o+ne\text{-}1{:}o+ne) = \langle o+ne,2\rangle$ ➔
- $Q(o+ne+1{:}o+(n+1)e) = \langle o+(n+1)e,2\rangle^{e/2}$
- Proof
- (2a) Show $Q(o+ne\text{-}1{:}o+ne) = \langle o+ne,2\rangle$ ➔ $Q(o+ne+1{:}o+ne+2) = \langle o+(n+1)e,2\rangle$
- (2b) Assume $o'+1 \leq e\text{-}2$: Show
- $Q(o+ne+o'{:}o+ne+o'+1) = \langle o+(n+1)e,2\rangle$ ➔ $Q(o+ne+o'+2{:}o+ne+o'+3) = \langle o+(n+1)e,2\rangle$

Thank You

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- Questions?
- Answers?☺