On Cameron-Liebler line classes with large parameter

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(joint work with Jeroen Demeyer, Klaus Metsch and Morgan Rodgers)

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Galois geometry

- $\mathrm{PG}(d, q)$: Projective space of dimension d over finite field $\mathrm{GF}(q)$: elements are subspaces of dimension at least 1 of the $d + 1$ dimensional vector space over $\mathrm{GF}(q)$.
- Analytic framework: coordinates, matrix groups etc.
- Sesquilinear and quadratic forms: totally isotropic elements of underlying vector space make a nice geometry: *classical polar space*.
- Finite simple groups of Lie type.

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A set $\mathcal{L}$ of lines of $\mathrm{PG}(3, q)$ is a *Cameron-Liebler* line class with parameter x if and only if $|\mathcal{L} \cap \mathcal{S}| = x$ for every spread $\mathcal{S}$ of $\mathrm{PG}(3, q)$.

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- Introduced in an attempt to classify collineation groups of $\mathrm{PG}(3,q)$ that have equally many point orbits and line orbits.
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Classical polar spaces

- $\theta_r(q) := \frac{q^{r+1}-1}{q-1}$

Definition

An x-tight set $\mathcal{L}$ of a finite classical polar space $\mathcal{P}$ of rank $r \geq 2$, is a set of $x\theta_{r-1}(q)$ points, such that

$$|P^{\perp} \cap \mathcal{L}| = \begin{cases} x\theta_{r-2}(q) + q^{r-1} & \text{if } P \in \mathcal{L} \\ x\theta_{r-2}(q) & \text{if } P \notin \mathcal{L}. \end{cases}$$

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Klein correspondence

A Cameron-Liebler line class of $\mathrm{PG}(3, q)$ with parameter x, is equivalent to an x-tight set of $\mathrm{Q}^+(5, q)$.

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Algebraic combinatorics

Theorem (Bamberg, Kelly, Law, Penttila)

Let A be the collinearity matrix of $\mathrm{Q}^+(5,q)$, and let $\mathcal{L}$ be an x-tight set with characteristic vector χ. Then

$$\chi - \frac{x}{q^2-1}\mathbf{j}$$

is an eigenvector of A with eigenvalue q^2-1, $\mathbf{j}$ the all one vector.

non-existence

Theorem (K. Metsch (2010))

A Cameron-Liebler line class in $\mathrm{PG}(3, q)$ *with parameter* x *does not exist for* $2 < x \leq q$.

Constructions

Constructions of Cameron-Liebler line classes:

- Bruen, Drudge: q odd, $x = \frac{q^2+1}{2}$
- Govaerts, Penttila: $q = 4$, $x \in \{4, 5\}$.

The quest for new examples

- $q = 2 \mod 3$, $x = \frac{(q+1)^2}{3}$
- $q = 3^h$, $x = \frac{(q^2-1)}{2}$
- For all examples, the group $C_3 : C_{q^2+q+1}$ is a subgroup of the automorphism group

Recent examples for $q \not\equiv 1 \mod 3$

Morgan Rodgers found examples for $q \leq 200$:

- $q \equiv 1 \mod 4$: $x = \frac{q^2-1}{2}$
- $q \equiv 2 \mod 4$: $x = \frac{(q+1)^2}{3}$

Using the group

- $G = C_{q^2+q+1}$
- orbits on points of $\mathrm{PG}(3, q)$: π_∞, $\{(1, 0, 0, 0)\}$, $q - 1$ orbits of length $q^2 + q + 1$
- orbits on lines of $\mathrm{PG}(3, q)$: lines through $(1, 0, 0, 0)$, lines in π_∞, $q^2 - 1$ orbits of length $q^2 + q + 1$.
- reconstruct the example, and investigate the intersection properties of the line class and the point orbits.

Some observations

- The $q-1$ point orbits are third degree surfaces in $\mathrm{PG}(3, q)$.
- The C_3 is generated by the Frobenius automorphism from $\mathbb{F}_{q^3} \rightarrow \mathbb{F}_q$.
- Some examples seems to have a larger automorphism group.

Bruen-Drudge construction

- Choose an elliptic quadric $\mathrm{Q}^-(3, q)$ in $\mathrm{PG}(3, q)$.
- There are $\frac{(q^2+1)q^2}{2}$ secant lines
- There are $q + 1$ tangent lines through each point of $\mathrm{Q}^-(3, q)$, choose half of them for each point
- the secant lines together with the chosen tangent lines is a Cameron-Liebler line class with parameter $x = \frac{q^2+1}{2}$.

Algebraic description

- We use $\mathbb{F}_{q^3}$ to represent $\mathrm{AG}(3, q)$.
- The non-trivial point orbits are now

$$\{\beta u^i \mid \beta \in \mathbb{F}_q \setminus \{0\}, i = 0 \dots q^2 + q\},$$

  where u is an element of order $q^2 + q + 1$ in $\mathbb{F}_{q^3}$.
- Notice: the Frobenius automorphism from $\mathbb{F}_{q^3} \to \mathbb{F}_q$ stabilizes the point orbits.

Combinatorics of the third degree surface

Suppose $q \neq 3^h$

- lines through 0: $q^2 + q + 1$
- lines at ∞: $q^2 + q + 1$
- lines meeting in 0 points: $\frac{q^2-q-2}{3}(q^2 + q + 1)$
- lines meeting in 1 point: $\frac{q^2-q-2}{2}(q^2 + q + 1)$
- lines meeting in 2 points: $(q + 1)(q^2 + q + 1)$
- lines meeting in 3 points: $\frac{q^2-q-2}{6}(q^2 + q + 1)$

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Combinatorics of the third degree surface

Suppose $q \neq 3^h$: all lines behave the same:

- 0 points of $\frac{q-2}{3}$ surfaces
- 1 point of $\frac{q-2}{2}$ surfaces
- 2 points of 1 surface
- 3 points of $\frac{q-2}{6}$ surfaces

Combinatorics of the third degree surface

Suppose $q = 3^h$: two line types:
Type (I): q^2 lines:

- 0 points of $\frac{q-3}{3}$ surfaces
- 1 point of $\frac{q-1}{2}$ surfaces
- 2 points of 1 surface
- 3 points of $\frac{q-3}{6}$ surfaces

Type (II): q lines:

- 0 points of $\frac{2q-3}{3}$ surfaces
- 3 points of $\frac{q}{3}$ surfaces

Final objectives

- describe infinite families of Cameron-Liebler line classes for $q = 2^h$, $x = \frac{(q+1)^2}{3}$
- describe infinite families of Cameron-Liebler line classes for $q = 3^h$, $x = \frac{q^2-1}{2}$
- investigate new examples: $q = 27$, $x = \frac{(q+1)^2}{2}$, this is probably also a member of an infinite family.
- describe more infinite families for $q = p^h$, $p \notin \{2, 3\}$.