Switching operations and binary codes of Hadamard matrices

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The matrix on the right is *normalized.*

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...but astronomical growth in the number of equivalence classes seems to occur.

Numbers of equivalence classes

n	num. equiv. classes	
4	1	
8	1	
12	1	Hadamard
16	5	Todd, M. Hall
20	3	M. Hall
24	60	Ito, Leon, Longyear; Kimura
28	487	Kimura
32	$\geq 13,561,612$	O.
36	$\geq 18,000,000$	O.
40	$\geq 3.66 \times 10^{11}$	Lam, Lam, Tonchev

Goal of this talk:

To propose some ideas for understanding and managing the combinatorial explosion using codes and switching operations.

3-normalization

If H is an $n \times n$ Hadamard matrix, $n > 2$, then rows 1–3 of H may be put in the form

$$\begin{array}{c} n/4 \quad\quad n/4 \quad\quad n/4 \quad\quad n/4 \\ \begin{pmatrix} 1 \ldots 1 & 1 \ldots 1 & 1 \ldots 1 & 1 \ldots 1 \\ 1 \ldots 1 & 1 \ldots 1 & - \cdots - & - \cdots - \\ 1 \ldots 1 & - \cdots - & 1 \ldots 1 & - \cdots - \end{pmatrix}. \end{array}$$

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$\Longrightarrow 4 \mid n$

The fourth row. . .

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$$\begin{array}{cccc} n/4 & n/4 & n/4 & n/4 \\ \left(\begin{array}{cccc} 1\ldots1 & 1\ldots1 & 1\ldots1 & 1\ldots1 \\ 1\ldots1 & 1\ldots1 & -\cdots- & -\cdots- \\ 1\ldots1 & -\cdots- & 1\ldots1 & -\cdots- \\ a & -b & -c & d \end{array}\right). \end{array}$$

Orthogonality $\Rightarrow$ vectors a, b, c, d have same number of 1s.

Type-j quadruples

Let $\#(r)$ denote numbers of 1s in vector r.
A set of four rows is a *type-j quadruple* if

- $\#(a) = j$, or
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Type-0 quadruples can exist only when $n = 4$ or $8 \mid n$.
(Orthogonality of rows 4 and 5 fails, otherwise.)

Switching type-0 quadruples

Consider two forms of type-0 quadruple,

$$\mathbf{Q} = \begin{pmatrix} 1\ldots1 & 1\ldots1 & 1\ldots1 & 1\ldots1 \\ 1\ldots1 & 1\ldots1 & -\cdots- & -\cdots- \\ 1\ldots1 & -\cdots- & 1\ldots1 & -\cdots- \\ 1\ldots1 & -\cdots- & -\cdots- & 1\ldots1 \end{pmatrix}$$

$$\mathbf{Q}' = \begin{pmatrix} -\cdots- & 1\ldots1 & 1\ldots1 & 1\ldots1 \\ -\cdots- & 1\ldots1 & -\cdots- & -\cdots- \\ -\cdots- & -\cdots- & 1\ldots1 & -\cdots- \\ -\cdots- & -\cdots- & -\cdots- & 1\ldots1 \end{pmatrix}$$

Then

$$\mathbf{H} = \begin{pmatrix} \mathbf{Q} \\ \mathbf{A} \end{pmatrix} \text{ is Hadamard } \iff \mathbf{H}' = \begin{pmatrix} \mathbf{Q}' \\ \mathbf{A} \end{pmatrix} \text{ is Hadamard}$$

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$\mathbf{H}$ and $\mathbf{H}'$ are generally not equivalent.

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n = 32 :

From the Sylvester equivalence class one obtains exactly 13,561,612 classes by switching. Additional classes are known, so this is only a lower bound.

Switching produces many new Hadamard matrices

n = 24 :

The 60 equivalence classes fall into nine switching classes of sizes

j	1	2	3	4	5	6	7	8	9	total
num.	8	8	17	10	5	5	5	1	1	60

As representatives of the nine classes we may take the eight matrices of the form $\begin{bmatrix} H_1 & H_1 \\ H_2 & -H^2 \end{bmatrix}$, with H_1 and H_2 Hadamard matrices of size 12, and the Paley matrix.

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To obtain a $\{0, 1\}$ matrix from H, normalize H; replace -1 with 0. Call the resulting matrix A.

Two codes: the *row code* $\mathcal{C}(H)$ and the *column code* $\mathcal{C}(H^{\mathsf{T}})$.

Properties of the code

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- If $n \equiv 8 \pmod{16}$ then $\mathcal{C}(H) = \mathcal{C}(H)^\perp$ (it's self-dual) $\Longrightarrow \dim \mathcal{C}(H) = n/2$. (From properties of invt. factors of H.)

Summary of codes, $n \equiv 0 \pmod 8$

$\mathcal{C}(H)$ and $\mathcal{C}(H^{\mathsf{T}})$ are

- doubly-even,
- self-orthogonal and possibly self-dual,
- always self-dual when $n \equiv 8 \pmod{16}$,
- of dimension at most $n/2$,
- of dimension at least $\lfloor \log_2 n \rfloor + 1$ (W.D. Wallis).

$n \equiv 0 \pmod 8$: weight-4 codewords

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$$
\begin{array}{c} \\ \\ i \\ j \\ k \\ l \\ \\ \\ \end{array}
\begin{array}{c} A \\ \begin{pmatrix} \\ \vdots \\ \\ 1\ldots0\ldots1\ldots0\ldots1\ldots0\ldots1\ldots0 \\ 1\ldots0\ldots1\ldots0\ldots0\ldots1\ldots0\ldots1 \\ 1\ldots0\ldots0\ldots1\ldots1\ldots0\ldots0\ldots1 \\ 1\ldots0\ldots0\ldots1\ldots0\ldots1\ldots1\ldots0 \\ \\ \vdots \\ \\ \end{pmatrix} \end{array}
\begin{array}{c} w \\ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{array}
$$

$n \equiv 0 \pmod 8$: weight-4 codewords

Consider the column code, $\mathcal{C}(H^{\mathsf{T}})$; regard codewords as column vectors.

Let $w \in \mathcal{C}(H^{\mathsf{T}})$ be a weight-4 codeword with support in rows i, j, k, l.

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$\Longrightarrow \{i, j, k, l\}$ is a type-0 quadruple.

(Follows from self-orthogonality of $\mathcal{C}(H^{\mathsf{T}})$ and orthogonality of w with every column of A.)

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Partial converse:

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If $\mathcal{C}(H^{\mathsf{T}})$ is self-dual, then w is in the code; otherwise it may or may not be.

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Why?

The columns of A generate $\mathcal{C}(H^{\mathsf{T}})$ and contain an even numbers of 1s in rows i, j, k, l. Hence w is orthogonal to every codeword in $\mathcal{C}(H^{\mathsf{T}})$.

$n \equiv 0 \pmod 8$: weight-4 codewords. Summary

If $n \equiv 8 \pmod{16}$ then $\mathcal{C}(H^{\mathsf{T}}) = \mathcal{C}(H^{\mathsf{T}})^{\perp}$. Hence there is a one-to-one correspondence between type-0 quadruples in H and weight-4 codewords in $\mathcal{C}(H^{\mathsf{T}})$.

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If $n \equiv 8 \pmod{16}$ then $\mathcal{C}(H^{\mathsf{T}}) = \mathcal{C}(H^{\mathsf{T}})^{\perp}$. Hence there is a one-to-one correspondence between type-0 quadruples in H and weight-4 codewords in $\mathcal{C}(H^{\mathsf{T}})$.

If $n \equiv 0 \pmod{16}$ then $\mathcal{C}(H^{\mathsf{T}}) \subseteq \mathcal{C}(H^{\mathsf{T}})^{\perp}$. In this case we must distinguish between type-0 quadruples that correspond to weight-4 codewords in $\mathcal{C}(H^{\mathsf{T}})$ and those that do not.

$n \equiv 0 \pmod{16}$: extremal examples

The code of the Sylvester matrix is the Reed-Müller code. For $n \geq 16$ it contains no weight-4 codewords, but Sylvester matrices contain numerous type-0 quadruples.

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Opposite extreme: matrices with self-dual codes. (There are two of size 16.)

Switching and codes

Let $\{i, j, k, l\}$ be a type-0 quadruple of H, and let w be the weight-4 word with support in rows i, j, k, l. Let H' be the Hadamard matrix that results from switching. Switching can be regarded as adding w to certain columns of the generator matrix A to produce a new generator matrix A'. Consequently

$$\langle \mathcal{C}(H^{\mathsf{T}}), w \rangle = \langle \mathcal{C}(H'^{\mathsf{T}}), w \rangle$$

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Three cases:

1. $w \in \mathcal{C}(H^{\mathsf{T}})$, $w \in \mathcal{C}(H'^{\mathsf{T}}) \Longrightarrow \mathcal{C}(H^{\mathsf{T}}) = \mathcal{C}(H'^{\mathsf{T}})$.
2. $w \in \mathcal{C}(H^{\mathsf{T}})$, $w \notin \mathcal{C}(H'^{\mathsf{T}}) \Longrightarrow \mathcal{C}(H'^{\mathsf{T}}) \subset \langle \mathcal{C}(H'^{\mathsf{T}}), w \rangle = \mathcal{C}(H^{\mathsf{T}})$; $\dim \mathcal{C}(H^{\mathsf{T}}) = 1 + \dim \mathcal{C}(H'^{\mathsf{T}})$. (Or the reverse: $H \leftrightarrow H'$.)
3. $w \notin \mathcal{C}(H^{\mathsf{T}})$, $w \notin \mathcal{C}(H'^{\mathsf{T}}) \Longrightarrow \dim \mathcal{C}(H^{\mathsf{T}}) = \dim \mathcal{C}(H'^{\mathsf{T}})$.

Switching and codes: $n \equiv 8 \pmod{16}$

Case (1) holds.
$\mathcal{C}(H^{\mathsf{T}})$ and $\mathcal{C}(H'^{\mathsf{T}})$ are the same self-dual code, which contains w.

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Conjecture: This is the only situation in which Case (1) holds.

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Conjecture: Either Case (2) or Case (3) holds.

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More specific conjecture:

- When $n \equiv 16 \pmod{32}$ Case (2) holds. Exactly one of $\mathcal{C}(H^{\mathsf{T}})$ and $\mathcal{C}(H'^{\mathsf{T}})$ contains w.

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More specific conjecture:

- When $n \equiv 16 \pmod{32}$ Case (2) holds. Exactly one of $\mathcal{C}(H^{\mathsf{T}})$ and $\mathcal{C}(H'^{\mathsf{T}})$ contains w.
- When $n \equiv 0 \pmod{32}$ either Case (2) or Case (3) holds. (It is possible that neither $\mathcal{C}(H^{\mathsf{T}})$ nor $\mathcal{C}(H'^{\mathsf{T}})$ contains w.)

Switching and codes: $n \equiv 8 \pmod{16}$

Further consequences:

- $\mathcal{C}(H^{\mathsf{T}})$ is invariant under switching.
- The set of type-0 quadruples in H is invariant under switching.
- Switching operations can be composed, and therefore form a group, the *switching group.*

$n \equiv 0 \pmod 8$: weight-4 codewords

Define $\mathcal{F}(H^{\mathsf{T}}) \subseteq \mathcal{C}(H^{\mathsf{T}})$ to be the subcode generated by the weight-4 codewords of $\mathcal{C}(H^{\mathsf{T}})$.
What are the possible structures of $\mathcal{F}(H^{\mathsf{T}})$?

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What are the possible structures of $\mathcal{F}(H^{\mathsf{T}})$?

Conway and Pless: Any doubly-even, self-orthogonal code generated by weight-4 codewords is a direct sum of three types of code:

- d_{2k}, $k \geq 2$,
- e_7, the even-weight subcode of the $[7, 4, 3]$ Hamming code,
- e_8, the $[8, 4, 4]$ extended Hamming code.

The code $d_{2k}\ldots$

$\ldots$ has length $2k$ and is generated by the $k-1$ row vectors

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & & & & & & \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots & 1 & 1 \end{bmatrix}$$

It contains $\binom{k}{2}$ codewords of weight 4.

The code e_7...

...has length 7 and is generated by the three row vectors

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

It contains seven codewords of weight 4.

The code e_8...

...has length 8 and is generated by the four row vectors

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

It contains 14 codewords of weight 4.

Restrictions on $\mathcal{F}(H^{\mathsf{T}})$

- The code of H_8 is e_8; e_8 does not appear in the code of any H_n for any other $n \equiv 8 \pmod{16}$.
- e_7 does not appear in the code of any H_n for $n = 24$ (Assmus and Key) or $n = 40$.

$n \equiv 8 \pmod{16}$: The switching group

The switching group is a subgroup acting on a given Hadamard matrix is a subgroup of the orthogonal group. It is the direct sum of the switching groups acting on the components d_{2k} and e_7 of which $\mathcal{F}(H^{\mathsf{T}})$ is composed.

On d_4, the group is a 2-element group. On d_{2k}, $k \geq 3$ the group has $2^{k-1}k!$ elements. The center is the set of elements that permute the rows of an even number of duads of d_{2k}. Since row permutation does not change the Hadamard equivalence class, we take the quotient by this center to obtain the group of moves that potentially do change the Hadamard equivalence class. This group is isomorphic to S_k.

Example: $n = 24$

Since $24 \equiv 8 \pmod{16}$,

- the code is self-dual,
- there is a one-to-one correspondence between weight-4 codewords and type-0 quadruples,
- switching does not change the code,
- the subcode $\mathcal{F}(H^{\mathsf{T}})$ must be a direct sum of codes d_{2k} and e_7 only; e_7 cannot occur in size 24.

Example: $n = 24$

There are nine self-dual, doubly-even codes of length 24 (Conway and Pless). Of these, three contain either e_7 or e_8 and are therefore not relevant to Hadamard matrices. The remaining six codes include the extended Golay code and five codes with $\mathcal{F} = d_{24}, 2d_{12}, 3d_8, 4d_6, 6d_4$. All six occur. (Assmus and Key)

Example: $n = 24$

The code of $H = \begin{bmatrix} H_1 & H_1 \\ H_2 & -H^2 \end{bmatrix}$ where H_1 and H_2 are 12×12 Hadamard matrices is the code containing d_{24}. The code of H^{T} may be any of the six codes. There are eight such transposed matrices H up to equivalence. The codes containing $4d_6$ and $6d_4$ each occur twice; the other four codes occur once. We have eight switching classes of sizes 8 (d_{24}), 8 ($2d_{12}$), 17 ($3d_8$), 10 and 5 ($4d_6$), 5 and 5 ($6d_4$), and 1 (the extended Golay code which, of course, has no weight-4 codewords).

The code of the Paley matrix is also the extended Golay code.

Example: $n = 16$

- Must consider non-self-dual codes as well.
- Switching changes the dimension of the code by ± 1.
- The smaller code is a subcode of the larger.
- Every self-orthogonal, doubly even code is a subcode of a self-dual, doubly-even code.
- The latter have been classified up to $n = 32$ by Conway and Pless. For $n = 16$ there are two self-dual codes, $e_8 \oplus e_8$ and $\langle d_{16} \cup \{(1010\ldots10)\}\rangle$.
- Each of the self-dual codes corresponds to one Hadamard matrix.

Example: $n = 16$

Mike Grilli (Research Experience for Undergraduates, Indiana University 2008) worked out a method for systematically identifying subcodes of self-dual codes that are potientially relevant to Hadamard matrices. For length 16, the subcodes of $e_8 \oplus e_8$ and $\langle d_{16} \cup \{(1010\ldots10)\}\rangle$ include

- one relevant subcode of dimension 7, which is common to the two self dual codes, and contains $d_8 \oplus d_8$,
- one relevant subcode of dimension 6, which contains $d_4 \oplus d_4 \oplus d_4 \oplus d_4$,
- one relevant subcode of dimension 5, the Reed-Müller code, which has no weight-4 codewords.