Computational Complexity Aspects of Graph Pebbling

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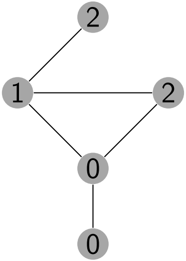
Graph Pebbling



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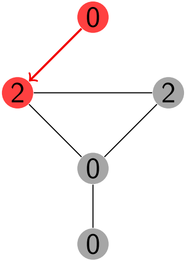

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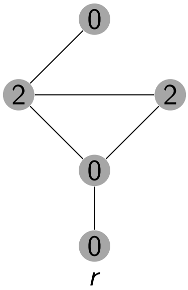

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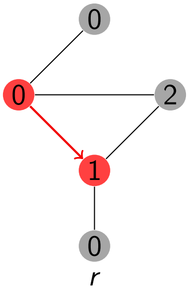

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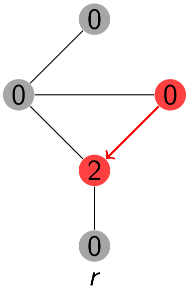

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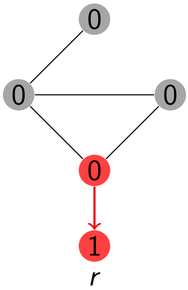

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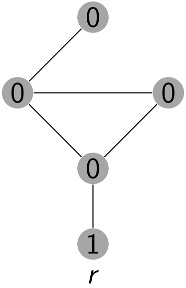

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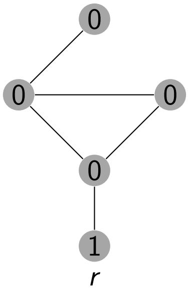

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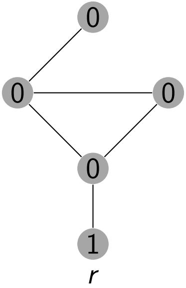

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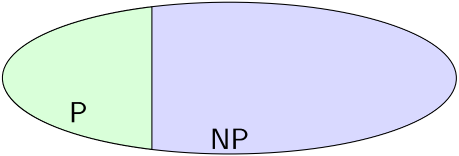
- Are there fast algorithms for this problem?
- Probably not: many problems are special cases of REACHABILITY

Computational Complexity: NP-completeness



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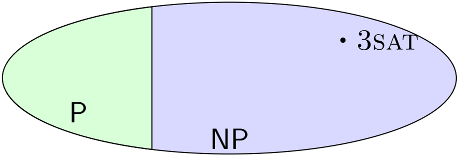

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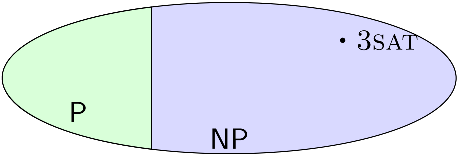

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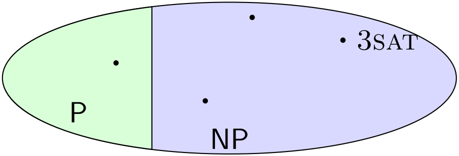

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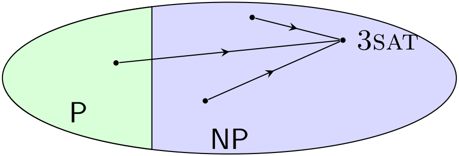

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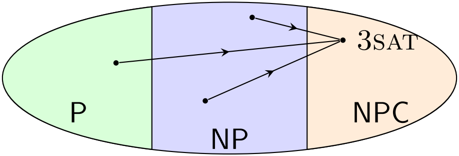

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Theorem (Cook; Levin)

3SAT *is NP-complete.*

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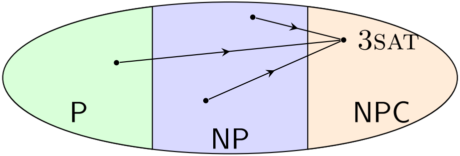

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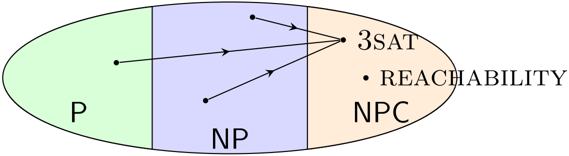

Fact

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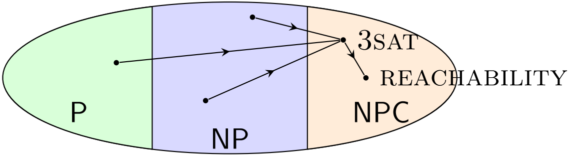

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There is a polynomial time reduction from 3SAT *to* REACHABILITY*.*

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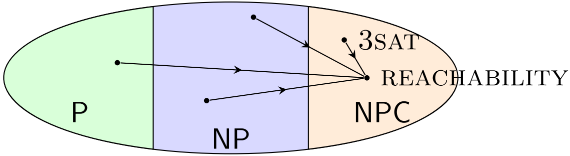

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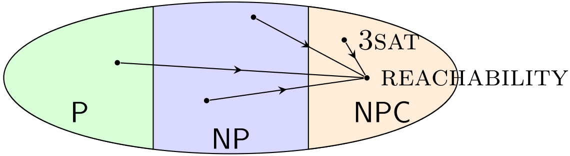

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Corollary (Hurlbert-Kierstead; Watson; Clark-Milans)

REACHABILITY *is NP-complete. If there is a polynomial time algorithm for* REACHABILITY, *then P=NP.*

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Proposition

There is a polynomial time algorithm to convert a 3CNF formula to an equivalent simple 3CNF formula.

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• 3SAT	• REACHABILITY

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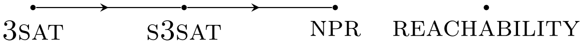

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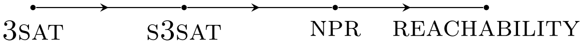

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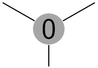

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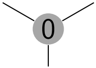

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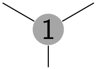

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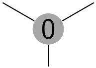

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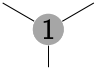

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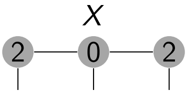

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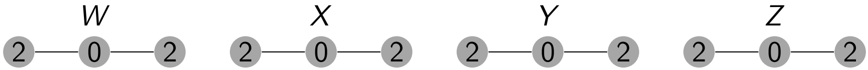

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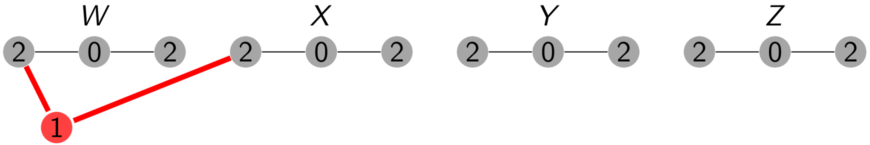

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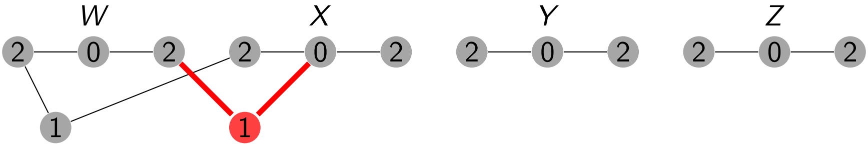

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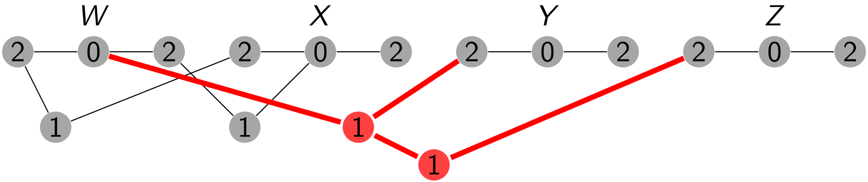

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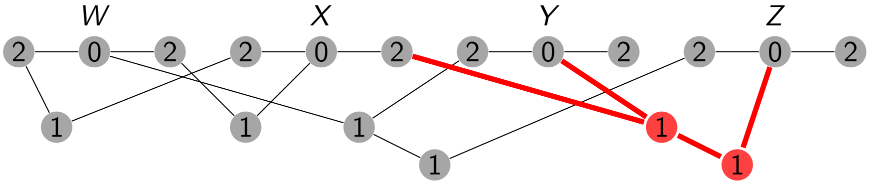

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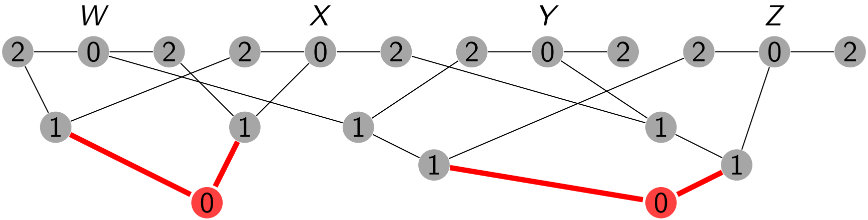

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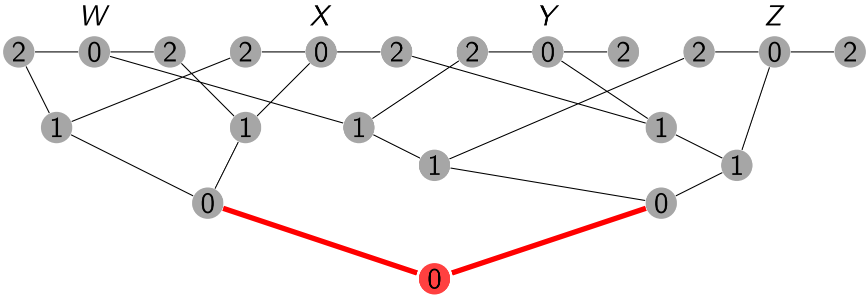

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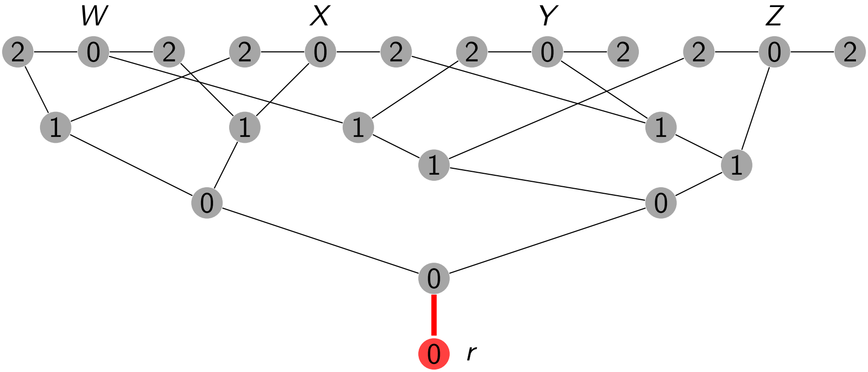

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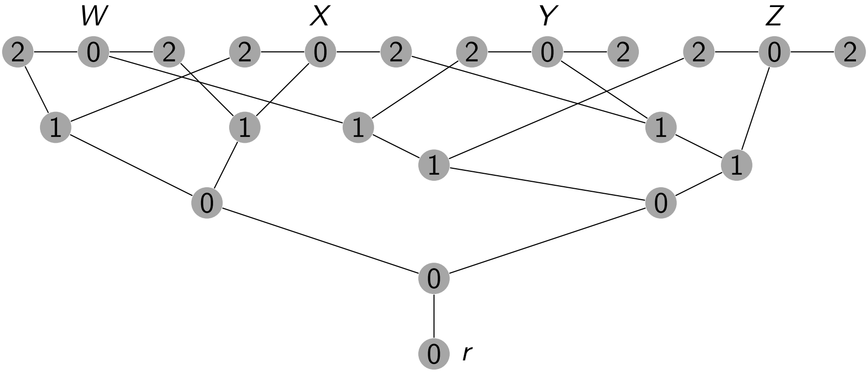

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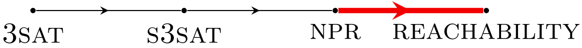

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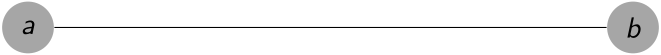

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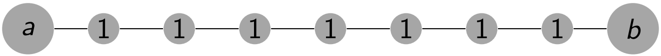

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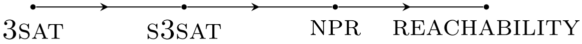

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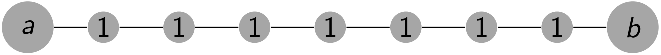

Theorem

REACHABILITY *is NP-complete even for bipartite graphs with* $\Delta(G) \leq 3$ *and at most 2 pebbles on each vertex.*

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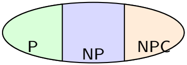
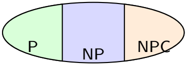
- $\pi(G)$: min k such that each dist. of k pebbles is solvable.
- PEBBLING-NUMBER: given G and k, is $\pi(G) \leq k$?

Beyond NP



- We usually think of NP as containing “hard problems”.

Beyond NP



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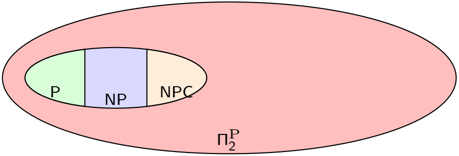

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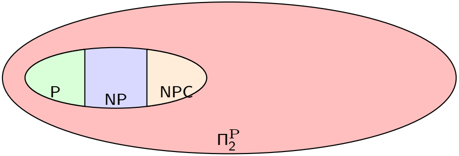

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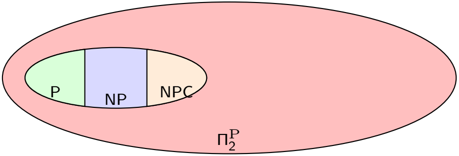

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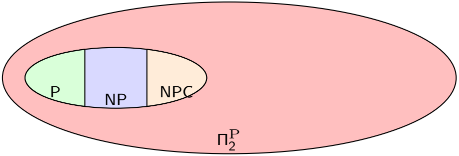

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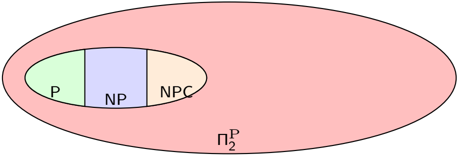

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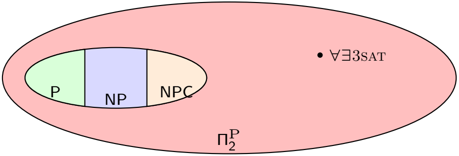

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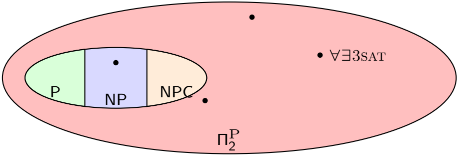

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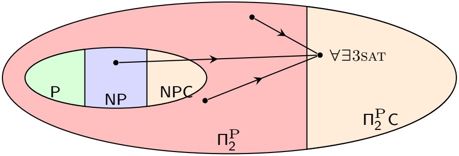

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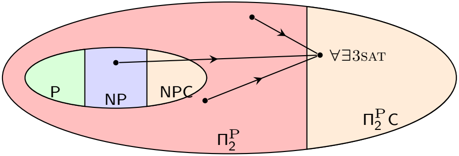

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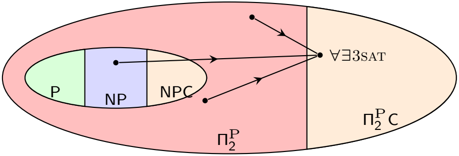

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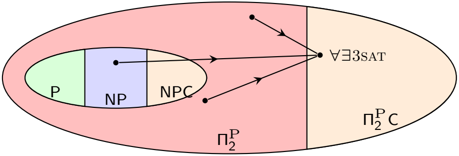

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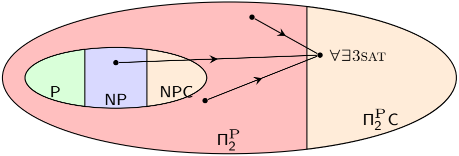

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- $\forall\exists 3\text{SAT}$ example is a “no” instance: if w is false, first two clauses are unsatisfiable.

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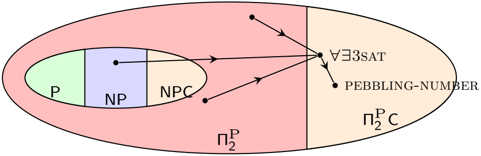

Theorem

There is a polynomial time reduction from ∀∃3SAT *to* PEBBLING-NUMBER.

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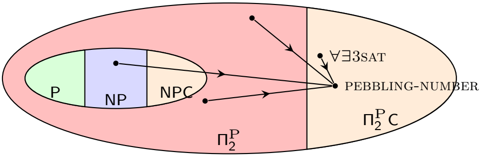

Theorem

There is a polynomial time reduction from ∀∃3SAT *to* PEBBLING-NUMBER.

Corollary

PEBBLING-NUMBER *is* Π_2^P*-complete.*

Complexity of Pebbling Number Variants

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[1] Vuong–Wyckoff, Sjöstrand

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- Approximation algorithms for $\pi(G)$.

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