

The Erdős-Ko-Rado Theorem for Permutations

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Intersecting Set Systems

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Answer must depend on n , for example, if $2k - t \geq n$ any two k -sets will be t -intersecting!

Examples

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The collection of all k -sets that contain a fixed t -set is called a *trivially t -intersecting k -set system*.

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- ▶ **Wilson 1984** For $n > (t+1)(k-t+1)$ the largest t -intersecting set system is a trivially t -intersecting set system.
- ▶ **Ahlsweede and Khachatryan 1997** Gave the maximal t -intersecting k -set system for all values of n .

Define the system of k -subsets from $\{1, \dots, n\}$

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If

$$(k - t + 1) \binom{2 + \frac{t-1}{i+1}}{k-t+1} < n < (k - t + 1) \binom{2 + \frac{t-1}{i}}{k-t+1},$$

then \mathcal{A}_i is the unique (up to a permutation on $\{1, \dots, n\}$) t -intersecting k -set system with maximal cardinality.

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There are clear candidates for the largest system:

$$\mathcal{F}_{i,j} = \{\sigma \in \mathcal{S}_n \text{ with } \sigma(i) = j\}$$

for any $i, j \in \{1, \dots, n\}$.

These are coset of the stabiliser of a point.

EKR for Permutations

Theorem (Cameron and Ku (2003), Larose and Malvenuto (2003), Godsil and Meagher (2007),)

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- ▶ Godsil and Meagher - used an argument with eigenvectors.

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(Have bound easily, uniqueness is the hard part.)

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Equitable partitions of permutations that fix 1 and permutations that don't fix 1.

By Hoffman's ratio bound we have

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How do we show that the $\mathcal{F}_{i,j}$ are all the maximum independent sets?

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- ▶ What is a maximum clique?

Cliques in $D_t(n)$

A set $H \subseteq S_n$ is called *sharply t -transitive* if for any (ordered?) t -set a_1, a_2, \dots, a_t and any other t -set b_1, b_2, \dots, b_t there exists a unique permutation $\pi \in H$ such that $\pi(a_i) = b_i$.

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Fix a t -set, each $h \in H$ maps it to a unique t -set so

$|H| = n(n-1) \cdots (n-t+1)$. If $\sigma, \pi \in H$ then $\sigma^{-1}\pi$ agree in fewer than t elements (else σ and π would map a t -set to the same t -set)

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- ▶ Don't have what happens for small values of n .

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Conjecture

For every n the largest t -intersecting permutation system is a \mathcal{F}_i for some i

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If $n \geq 2t + 1$ then \mathcal{F}_0 is the largest system.