Crossings, Colorings, and Cliques

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Joint with Mike Albertson and Jacob Fox.

On Graphs with Crossings, CanaDAM
28 May 2009

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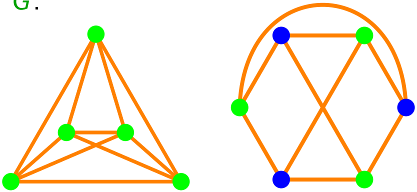

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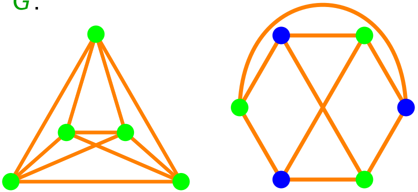

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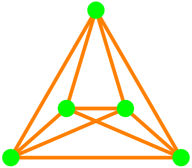

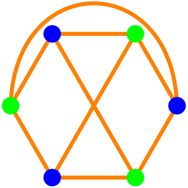

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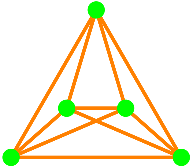

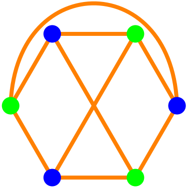

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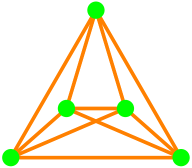

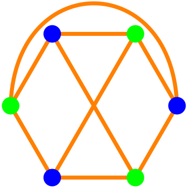

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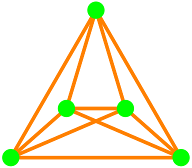

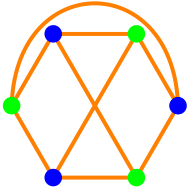

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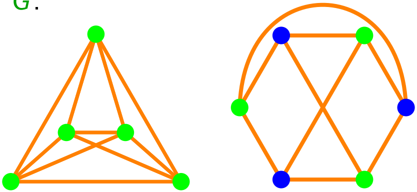

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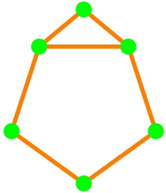

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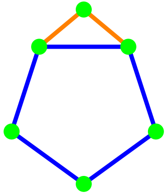

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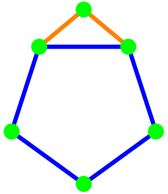

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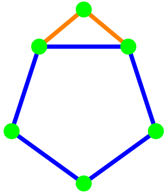

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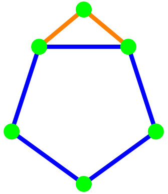

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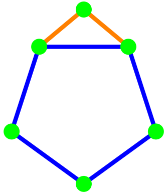

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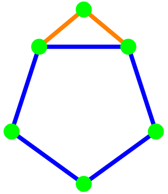

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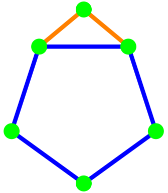

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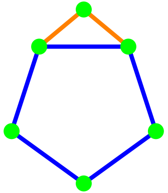

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Hence $m \geq 3n+1$ and $\mathrm{cr}(G) \geq m - (3n-6) \geq 7$. ■

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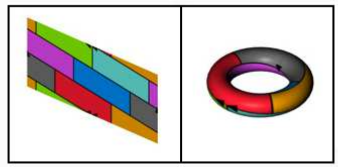

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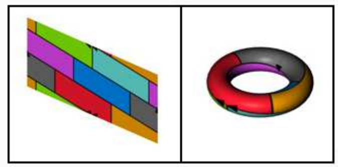

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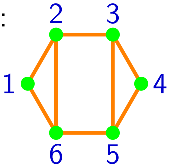

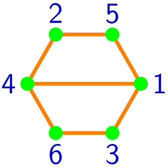

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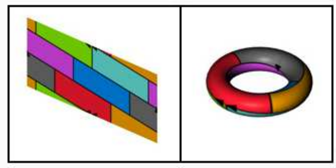

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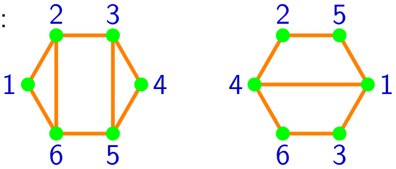

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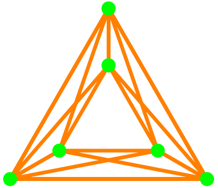

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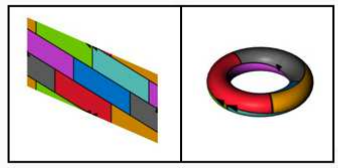

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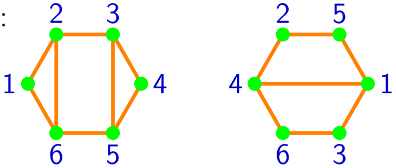

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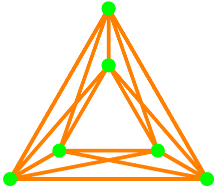

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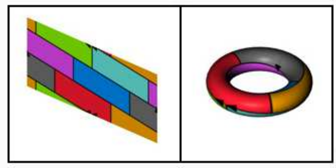

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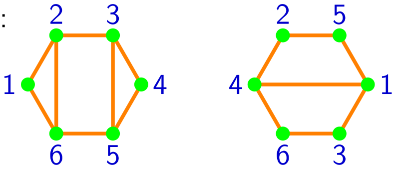

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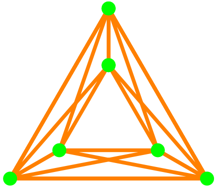

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If so, what are the extremal graphs?

Easier proofs of harder results

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Thm. [Kostochka-Stiebitz '96] If G is r-critical and $G \neq K_r$ and $n \neq 2r-1$, then

$$m \geq \frac{r-1}{2}n + r - 3.$$

Proving Albertson's Conjecture (for lots more cases)

Crossing Lemma [Leighton; Ajtai et. al. '82] If $m \geq 4n$, then

$$\mathrm{cr}(G) \geq \frac{1}{64}\frac{m^3}{n^2}.$$

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Thm. Albertson's Conjecture is true for $r \leq 12$.