Oligomorphic permutation groups: growth rates and algebras

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The definition

Let G be a permutation group on an infinite set Ω. Then G has a natural induced action on the set of all n-tuples of elements of Ω, or on the set of n-tuples of distinct elements of Ω, or on the set of n-element subsets of Ω. It is easy to see that if there are only finitely many orbits on one of these sets, then the same is true for the others.

We say that G is *oligomorphic* if it has only finitely many orbits on Ω^n for all natural numbers n.

We denote the number of orbits on all n-tuples, resp. n-tuples of distinct elements, n-sets, by $F_n^*(G)$, $F_n(G)$, $f_n(G)$ respectively.

Examples, 1

Let S be the symmetric group on an infinite set X. Then S is oligomorphic and

- $F_n(S) = f_n(S) = 1$,
- $F_n^*(S) = B(n)$, the nth *Bell number* (the number of partitions of a set of size n.

Let $A = \mathrm{Aut}(\mathbb{Q}, <)$, the group of order-preserving permutations of $\mathbb{Q}$. Then A is oligomorphic and

- $f_n(A) = 1$;
- $F_n(A) = n!$;
- $F_n^*(A)$ is the number of *preorders* of an n-set.

Examples, 2

Consider the group S^r acting on the disjoint union of r copies of X.

- $F_n(S^r) = r^n$;
- $f_n(S^r) = \binom{n+r-1}{r-1}$.

Consider S^r acting on Ω^r. Then $F_n^*(S^r) = B(n)^r$. From this we can find $F_n(S^r)$ by inversion:

$$F_n(G) = \sum_{k=1}^{n} s(n,k) F_k^*(G)$$

for any oligomorphic group G, where $s(n,k)$ is the signed *Stirling number* of the second kind.

For A^2 acting on $\mathbb{Q}^2$, $f_n(A^2)$ is the number of zero-one matrices (of unspecified size) with n ones and no rows or columns of zeros.

Examples, 3

Let $G = S \,\mathrm{Wr}\, S$, the wreath product of two copies of S. Then $F_n(G) = B(n)$ and $f_n(G) = p(n)$, the number of partitions of n.

Let $G = S_2 \,\mathrm{Wr}\, A$, where S_2 is the symmetric group of degree 2. Then $f_n(G)$ is the nth *Fibonacci number*.

Examples, 4

There is a unique *countable random graph* R: that is, if we choose a countable graph at random (edges independent with probability $\frac{1}{2}$, then with probability 1 it is isomorphic to R.

- R is *universal*, that is, it contains every finite or countable graph as an induced subgraph;
- R is *homogeneous*, that is, any isomorphism between finite induced subgraphs of R can be extended to an automorphism of R.

If $G = \mathrm{Aut}(R)$, then $F_n(G)$ and $f_n(G)$ are the numbers of labelled and unlabelled graphs on n vertices.

Connection with model theory, 1

If a set of sentences in a first-order language has an infinite model, then it has arbitrarily large infinite models. In other words, we cannot specify the cardinality of an infinite structure by first-order axioms.

Cantor proved that a countable dense total order without endpoints is isomorphic to $\mathbb{Q}$. Apart from countability, the conditions in this theorem are all first-order sentences.

What other structures can be specified by countability and first-order axioms? Such structures are called *countably categorical*.

Connection with model theory, 2

In 1959, the following result was proved independently by Engeler, Ryll-Nardzewski and Svenonius:

Theorem 1. *A countable structure M over a first-order language is countably categorical if and only if* $\mathrm{Aut}(M)$ *is oligomorphic.*

In fact, more is true: the *types* over the theory of M are all realised in M, and the sets of n-tuples which realise the n-types are precisely the orbits of $\mathrm{Aut}(M)$ on M^n.

Growth of $(f_n(G))$, 1

Several things are known about the behaviour of the sequence $(f_n(G))$:

- it is non-decreasing;
- either it grows like a polynomial (that is, $an^k \le f_n(G) \le bn^k$ for some $a, b > 0$ and $k \in \mathbb{N}$), or it grows faster than any polynomial;
- if G is *primitive* (that is, it preserves no non-trivial equivalence relation on Ω), then either $f_n(G) = 1$ for all n, or $f_n(G)$ grows at least exponentially;
- if G is *highly homogeneous* (that is, if $f_n(G) = 1$ for all n), then either there is a linear or circular order on Ω preserved or reversed by G, or G is *highly transitive* (that is, $F_n(G) = 1$ for all n).
- There is no upper bound on the growth rate of $(f_n(G))$.

Growth of $(f_n(G))$, 2

Examples suggest that much more is true. For any reasonable growth rate, appropriate limits should exist:

- for polynomial growth of degree k, $\lim(f_n(G)/n^k)$ should exist;
- for fractional exponential growth (like $\exp(n^c)$), $\lim(\log\log f_n(G)/\log n)$ should exist;
- for exponential growth, $\lim(\log f_n(G)/n)$ should exist;

and so on.

I do not know how to prove any of these things; and I do not know how to formulate a general conjecture.

A Ramsey-type theorem

Theorem 2. *Let X be an infinite set, and suppose that the n-element subsets of Ω are coloured with r different colours (all of which are used). Then there is an ordering $(c_1, \ldots, c_r)$ of the colours, and infinite subsets $Y_1, \ldots, Y_r$ of X, such that, for $i = 1, \ldots, r$, the set Y_i contains an n-set of colour c_i but none of colour c_j for $j > i$.*

The existence of Y_1 is the classical theorem of Ramsey.

There is a finite version of the theorem, and so there are corresponding 'Ramsey numbers'. But very little is known about them!

Monotonicity

Corollary 3. *The sequence $(f_n(G))$ is non-decreasing.*

Proof. Let $r = f_n(G)$, and colour the n-subsets with r colours according to the orbits. Then by the Theorem, there exists an $(n+1)$-set containing a set of colour c_i but none of colour c_j for $j > i$. These $(n+1)$-sets all lie in different orbits; so $f_{n+1}(G) \geq r$. □

There is also an algebraic proof of this corollary. We'll discuss this later.

A graded algebra, 1

Let $\binom{\Omega}{n}$ denote the set of n-subsets of Ω, and V_n the vector space of functions from $\binom{\Omega}{n}$ to $\mathbb{C}$.

We make $\mathcal{A} = \bigoplus_{n\geq 0} V_n$ into an algebra by defining, for $f \in V_n$, $g \in V_m$, the product $fg \in V_{n+m}$ by

$$(fg)(K) = \sum_{M \in \binom{K}{m}} f(M)g(K \setminus M)$$

for $K \in \binom{\Omega}{m+n}$, and extending linearly.

$\mathcal{A}$ is a commutative and associative graded algebra over $\mathbb{C}$, sometimes referred to as the *reduced incidence algebra* of finite subsets of Ω.

A graded algebra, 2

Now let G be a permutation group on Ω, and let V_n^G denote the set of fixed points of G in V_n. Put

$$\mathcal{A}[G] = \bigoplus_{n\geq 0} V_n^G,$$

a graded subalgebra of $\mathcal{A}$.

If G is oligomorphic, then the dimension of V_n^G is $f_n(G)$, and so the Hilbert series of the algebra $\mathcal{A}[G]$ is the ordinary generating function of the sequence $(f_n(G))$.

What properties does this algebra have?

Note that it is not usually finitely generated since the growth of $(f_n(G))$ is polynomial only in special cases.

A non-zero-divisor

Let e be the constant function in V_1 with value 1. Of course, e lies in $\mathcal{A}[G]$ for any permutation group G.

Theorem 4. *The element e is not a zero-divisor in $\mathcal{A}$.*

This theorem gives another proof of the monotonicity of $(f_n(G))$. For multiplication by e is a monomorphism from V_n^G to V_{n+1}^G, and so $f_{n+1}(G) = \dim v_{n+1}^G \geq \dim V_n^G = f_n(G)$.

An integral domain

If G has a finite orbit Δ, then any function whose support is contained in Δ is nilpotent.

The converse, a long-standing conjecture, has recently been proved by Maurice Pouzet:

Theorem 5. *If G has no finite orbits on Ω, then $\mathcal{A}[G]$ is an integral domain.*

Consequences

Pouzet's Theorem has a consequence for the growth rate:

Theorem 6. *If G is oligomorphic, then*

$$f_{m+n}(G) \geq f_m(G) + f_n(G) - 1.$$

Proof. Multiplication maps $V_m^G \otimes V_n^G$ into V_{m+n}^G; by Pouzet's result, it is injective on the projective Segre variety, and a little dimension theory gets the result. □

It seems very likely that better understanding of the algebra $\mathcal{A}[G]$ would have further implications for growth rate.

Brief sketch of the proof

Let $\mathcal{F}$ be a family of subsets of Ω. A subset T is *transversal* to $\mathcal{F}$ if it intersects each member of $\mathcal{F}$. The *transversality* of $\mathcal{F}$ is the minimum cardinality of a transversal.

A lemma due to Peter Neumann shows that, if G has no finite orbits on Ω, then any orbit of G on finite sets has infinite transversality.

Pouzet shows that, if $f \in V_m$ and $g \in V_n$ satisfy $fg = 0$, then the transversality of $\mathrm{supp}(f) \cup$

$\mathrm{supp}(g)$ is finite, and is bounded by a function of m and n. (Here $\mathrm{supp}(f)$ denotes the support of f.)

These two results clearly conflict with each other.

Comments

Here is Pouzet's theorem again:

Theorem 7. *If $f \in V_m$ and $g \in V_n$ satisfy $fg = 0$, then the transversality of* $\mathrm{supp}(f) \cup \mathrm{supp}(g)$ *is finite, and is bounded by a function of m and n.*

The proof of this makes it clear that it is another kind of 'Ramsey theorem'. If $\tau(m,n)$ denotes the smallest t such that the transversality is at most t, then we have the interesting problem of finding $\tau(m,n)$. Pouzet shows that $\tau(m,n) \geq (m+1)(n+1) - 1$. On the other hand, the upper bounds coming from his proof are really astronomical!