Discrete Geometry and Word Combinatorics

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From discrete geometry to word combinatorics... 

...via tilings
A **polyomino** is an edge connected and simply connected finite union of unit squares.
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How to code such an object?
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How to code such an object?

The Freeman code gives a representation of the boundary of a given pixellised object.
The **Freeman code** gives the segment directions of the contour of a bounded connected shape.

\[ w = 10010\bar{1}0\bar{1}0\bar{0}\bar{1}\bar{0}1\bar{0} \]
A word over the alphabet \( \{0, 1, \bar{0}, \bar{1}\} \) is a boundary word if it codes the boundary of a polyomino.

A polyomino is determined by the conjugation class of its boundary word.
Boundary words

A word over the alphabet \( \{0, 1, \bar{0}, \bar{1}\} \) is a boundary word if it codes the boundary of a polyomino.

A polyomino is determined by the conjugation class of its boundary word

Some questions...

- Can we recognize efficiently whether a word \( w \) is a boundary word?
- What can be said on the corresponding polyomino? Can we detect geometric properties like convexity?
- Does it tile the plane?
Self-avoiding paths

Let $w \in \{0, 1, \bar{0}, \bar{1}\}^n$. Can we recognize efficiently whether a word $w$ is a boundary word?

Theorem [S. Brlek-M. Koskas-X. Provençal '09]

- It is possible to recognize in $O(n)$ whether $w$ codes a self-avoiding path (simple curve)
- It is possible to determine in $O(n)$ whether $w$ is a boundary word

The idea is to use as data structure a radix tree with a binary encoding of locations.
Convexity and Lyndon words

Can we detect geometric properties like digital convexity?
Can we detect geometric properties like *digital convexity*?
Convexity and Lyndon words

Can we detect geometric properties like digital convexity?

Theorem [S. Brlek-J.-O. Lachaud-X. Provençal-C. Reutenauer ’09]

A word \( w \) is NW-convex if and only if its unique factorization in nondecreasing Lyndon words is composed of primitive Christoffel words
A tiling of $\mathbb{Z}^2$ by a set of polyominoes is a partition of $\mathbb{Z}^2$ into translates of the polyominoes of $\mathcal{T}$.

**Domino problem [R. Berger ’66, S. W. Golomb ’70]**

Given a set of polyominoes $\mathcal{T}$, it is **undecidable** to know whether it tiles $\mathbb{Z}^2$. 
Tiling of the plane by one polyomino

Theorem [H.A.G. Wijshoff-J. van Leuwen ’84]

A single polyomino that tiles the plane by translation tiles it biperiodically.

The tiling problem by translation is decidable for a single polyomino.
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Theorem [D. Beauquier-M. Nivat ’91]
A single polyomino tiles the plane by translation if and only if it is a pseudo-hexagon or a pseudo-square.
Exact polyominoes

Let $W = w_1 \cdots w_n \in \{0, 1, \bar{0}, \bar{1}\}^*$. We go from the free monoid on 4 letters to the free group on 2 letters

$$\bar{0} = 0^{-1}, \quad \bar{1} = 1^{-1}$$
Let $W = w_1 \cdots w_n \in \{0, 1, \overline{0}, \overline{1}\}^*$. We go from the free monoid on 4 letters to the free group on 2 letters

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$$u = 0 \ 0 \ 1 \ 0 \ \overline{1} \ 0 \ 1$$

$$u^{-1} = \overline{1} \ \overline{0} \ 1 \ \overline{0} \ \overline{1} \ \overline{0} \ \overline{0}$$

**Remark:** For an algebraic formalization of boundary words, see e.g. [Thurston-Conway-Lagarias] for tiling problems in finite domains.
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\[
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**Pseudo-hexagon**
\[
w \equiv XYZX^{-1}Y^{-1}Z^{-1}
\]

**Pseudo-square**
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**Theorem [D. Beauquier-M. Nivat '91]**

A polyomino \( P \) with boundary word \( b(P) \) tiles the plane if and only if there exist three words \( X, Y, Z \in \{0, 1, \bar{0}, \bar{1}\}^* \) such that

\[
XYZ\bar{X}\bar{Y}\bar{Z} \in b(P)
\]

where \( b(P) \) stands for the conjugation class of all boundary words of \( P \).
Exact polyominoes

Let $W = w_1 \cdots w_n \in \{0, 1, \bar{0}, \bar{1}\}^*$. We go from the free monoid on 4 letters to the free group on 2 letters

$$\bar{0} = 0^{-1}, \quad \bar{1} = 1^{-1}$$

**Theorem [S. Brlek-J.-M. Fédou-X. Provençal ’08]**

Let $w$ be the boundary word of a polyomino $P$ of length $n$. To determine whether $P$ tiles the plane can be tested in $O(n(\log n)^3))$.

To determine whether $P$ is a pseudo-square can be tested in $O(n)$. 
From tilings to discrete geometry

\[ \mathbb{Z}^2 \rightarrow \text{Honeycomb/hexagonal and triangular lattice polyomino} \rightarrow \text{polyamond (diamond)} \]

Lozenge tiling model/ Dimers on the honeycomb graph/ Perfect matching of a bipartite planar graph
Let $a, b, c$ be strictly positive real numbers

The (standard) arithmetic discrete plane $\mathcal{P}_{((a,b,c),h)}$ is defined as

$$\mathcal{P}_{((a,b,c),h)} = \{(p, q, r) \in \mathbb{Z}^3 \mid 0 \leq ap + bq + cr + h < a + b + c\}.$$
Let $a, b, c$ be strictly positive real numbers.

The (standard) arithmetic discrete plane $\mathcal{P}_{(a,b,c),h}$ is defined as

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We consider the stepped plane $\mathcal{P}_{((a,b,c),h)}$ defined as the set of facets of integer translates of unit cubes with vertices in $\mathcal{P}_{((a,b,c),h)}$.
From discrete planes to word combinatorics

We define a patch of a discrete plane as an edge connected and simply connected union of faces.

How to code such an object?

We can code

- the faces of a patch
- the boundary of a patch
Coding faces
An arithmetic discrete plane can be coded as

```
1 2 1 2 1 2 3 1 2 1 2 3 1 3
3 1 3 1 2 1 2 3 1 2 1 2 1 2
2 1 2 3 1 2 1 2 3 1 3 1 2 1
1 2 1 2 3 1 3 1 2 1 2 3 1 2
3 1 2 1 2 1 2 3 1 2 1 2 3 1
2 3 1 3 1 2 1 2 3 1 2 1 2 1
1 2 1 2 3 1 2 1 2 3 1 3 1 2
3 1 2 1 2 3 1 3 1 2 1 2 3 1
```
A first dynamical description

Let \((0, 1^*), (0, 2^*),\) and \((0, 3^*)\) be the three following faces:

Let \(\vec{x} \in \mathbb{Z}^3\). One sets \((\vec{x}, i^*) = \vec{x} + (\vec{0}, i^*)\).

\[
\mathcal{P}_{(\vec{n}, h)} = \{ \vec{x} \in \mathbb{Z}^3 \mid 0 \leq \langle \vec{x}, \vec{n} \rangle + h < ||\vec{n}||_1 \}
\]

The face \((\vec{x}, i^*)\) belongs to \(\mathcal{P}_{(\vec{n}, h)}\) if and only if

\[
0 \leq \langle \vec{x}, \vec{n} \rangle + h < \langle \vec{e}_i, \vec{n} \rangle
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\[
0 \leq \langle \vec{x}, \vec{n} \rangle + h < \langle \vec{e}_i, \vec{n} \rangle
\]

\[\Rightarrow\] Coding of a dynamical system given by a \(\mathbb{Z}^2\)-action by rotations on \(\mathbb{R}/\mathbb{Z}\)
Two-dimensional Sturmian words

The face $(\vec{x}, i^*)$ belongs to $\Psi(\vec{n}, h)$ if and only if

$$\langle \vec{x}, \vec{n} \rangle + h \in [0, \langle \vec{e}_i, \vec{n} \rangle)$$

We have a coding of a dynamical system given by a $\mathbb{Z}^2$-action by rotations on $\mathbb{R}/\mathbb{Z}$.

We deduce information on the

- number of configurations/factors of a given size (enumeration)
- frequencies (probabilities)

by classical methods from symbolic dynamics [B.-Vuillon]

Arithmetic discrete planes with the same normal vector have the same language.
Some classical problems in discrete geometry

We would like to be able to

- **Generate** discrete planes
- **Recognize** discrete planes: given a set of points in $\mathbb{Z}^3$, is it contained in an arithmetic discrete plane?

~~> Hierarchical structure/substitution rules
Generalized substitutions

We would like to give a description of $P(\vec{n}, h)$ with respect to a multidimensional continued fraction algorithm

\[
\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = M_1 \cdots M_k \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} = M_1 \cdots M_k \vec{n}_k
\]

where the $M_i \in SL(3, \mathbb{N})$. 
Generalized substitutions

We would like to give a description of $\mathcal{P}(\vec{n}, h)$ with respect to a multidimensional continued fraction algorithm

$$\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = M_1 \cdots M_k \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} = M_1 \cdots M_k \vec{n}_k$$

where the $M_i \in SL(3, \mathbb{N})$.

- Take a matrix $M \in SL(3, \mathbb{N})$. We want to find an algorithmic way to go from $\mathcal{P}_{M\vec{n}, h}$ to $\mathcal{P}_{\vec{n}, h}$,

that is, to associate sets of faces of each plane in a one-to-one correspondence.
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- Take a matrix $M \in SL(3, \mathbb{N})$. We want to find an algorithmic way to go from $P_{M\vec{n}, h}$ to $P_{\vec{n}, h}$, that is, to associate sets of faces of each plane in a one-to-one correspondence

We use the fact that

$$\langle \vec{x}, M \vec{n} \rangle = \langle t M \vec{x}, \vec{n} \rangle$$
Generalized substitutions

The face \((\vec{y}, j^*)\) belongs to \(P(\vec{n}, h)\) if and only if

\[
0 \leq \langle \vec{y}, \vec{n} \rangle + h < \langle \vec{e}_j, \vec{n} \rangle
\]
Generalized substitutions

The face $(\vec{y}, j^*)$ belongs to $\mathcal{P}(\vec{n}, h)$ if and only if

$$0 \leq \langle \vec{y}, \vec{n} \rangle + h < \langle \vec{e}_j, \vec{n} \rangle$$

Assume $(\vec{y}, j^*) \in \mathcal{P}_M \vec{n}, h$. One has

$$0 \leq \langle \vec{y}, M \vec{n} \rangle + h = \langle t^M \vec{y}, \vec{n} \rangle + h < \langle \vec{e}_j, M \vec{n} \rangle = \langle t^M \vec{e}_j, \vec{n} \rangle$$
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Assume \((\vec{y}, j^*) \in \mathcal{P}_{M \vec{n}, h}\). One has

\[
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\]

We use the fact that \(M \in SL(3, \mathbb{N})\). Let \(i\) such that \(M_{ji} \neq 0\). Write

\[
^t M \vec{e}_j = \vec{P} + \vec{e}_i + \vec{S},
\]

with \(\vec{P}, \vec{S}\) having nonnegative entries. There exists \(\vec{S}\) such that

\[
0 \leq \langle ^t M \vec{y} - \vec{S}, \vec{n} \rangle + h < \langle \vec{e}_i, \vec{n} \rangle,
\]

which implies

\[
(^t M \vec{y} - \vec{S}, i^*) \in \mathcal{P}_{\vec{n}, h}.
\]
Generalized substitutions

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which implies

\[ (t^M \vec{y} - \vec{S}, i^*) \in \mathcal{P}_{\vec{n}, h}. \]

We have associated with a face \((\vec{y}, j^*) \in \mathcal{P}_{M\vec{n}, h}\) a face \((\vec{x}, i^*) \in \mathcal{P}_{\vec{n}, h}\) with

\[ m_{ji} \neq 0 \quad \text{and} \quad \vec{x} = t^M \vec{y} - \vec{S} \]
Generalized substitutions

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\[ m_{ji} \neq 0 \quad \text{and} \quad \vec{x} = tM \vec{y} - \vec{S} \]

\[ tM \vec{e}_j = \vec{P} + \vec{e}_i + \vec{S}. \]

\[ \vec{y} = (tM)^{-1}(\vec{x} + \vec{S}) \quad \leadsto \quad \vec{x} = tM \vec{y} - \vec{S} \]

- This map is onto but not one-to-one. Let us take its "inverse".
- Let us formalize

\[ tM \vec{e}_j = \vec{P} + \vec{e}_i + \vec{S} \]

We consider a substitution whose incidence matrix is given by \(tM\).
Let $\sigma$ be a substitution on $\mathcal{A}$.

**Example:**

$$\sigma(1) = 12, \quad \sigma(2) = 13, \quad \sigma(3) = 1.$$ 

The **incidence matrix** $M_\sigma$ of $\sigma$ is defined by

$$M_\sigma = (|\sigma(j)|_i)_{(i,j) \in \mathcal{A}^2},$$

where $|\sigma(j)|_i$ counts the number of occurrences of the letter $i$ in $\sigma(j)$.

**Unimodular substitution**

$$\det M_\sigma = \pm 1$$
**Generalized substitutions**

**Abelianisation**

Let $d$ be the cardinality of $A$. Let $\vec{l} : A^* \to \mathbb{N}^d$ be the abelinisation map

$$\vec{l}(w) = t(|w|_1, |w|_2, \cdots, |w|_d).$$

**Generalized substitutions** [P. Arnoux-S. Ito][H. Ei]

Let $\sigma$ be a unimodular substitution.

$$E_1^*(\sigma)(\vec{x}, i^*) = \sum_{j \in A} \sum_{P, \sigma(j) = PiS} \left( M_{\sigma}^{-1} \left( \vec{x} + \vec{l}(S) \right), j^* \right).$$
Theorem [P. Arnoux-S. Ito, B.-Th. Fernique]

Let $\sigma$ be a unimodular substitution. Let $\vec{n} \in \mathbb{R}^d_+$ be a positive vector. The generalized substitution $E_1^*(\sigma)$ maps without overlaps the stepped plane $P_{\vec{n},h}$ onto $P_{tM_\sigma \vec{n},h}$.

The generalized substitution $E_1^*(\sigma)$ maps without overlaps stepped surfaces onto stepped surfaces.
Generalized substitutions
Some iterations
Generation
Generation
Generation
Generation
Generation
Consider a discrete plane whose normal vector is a left eigenvector of $M_\sigma$. Let $U$ be the unit cube at the origin. By considering the iterates of $U$ under the action of a generalized substitution, are we able to generate the whole discrete plane?
Recognition

One has

\[ E_1^\ast (\sigma \circ \tau) = E_1^\ast (\tau) \circ E_1^\ast (\sigma) \]

We can substitute and desubstitute

One gets a recognition algorithm inspired by the Sturmian case based on Brun’s algorithm [B.-Fernique]
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Changing the order of letters

1 → 12, 2 → 23, 3 → 123

1 → 12, 2 → 32, 3 → 231
Let \( \mathcal{U} \) stand for the unit cube.
Assume the sequence of patterns

\[
(E_1^*(\sigma))^n(\mathcal{U})
\]

generates the whole plane \( \mathcal{P}_{(\vec{n},h)} \)

What is the shape of these patterns?

**Theorem [H. Ei]** Let \( \sigma \) be an invertible three-letter substitution. The boundary of \( E_1^*(\sigma)(\mathcal{U}) \) is given by \( \sigma^{-1} \), the mirror image of the inverse of \( \sigma \)

**Theorem [B.,A. Lacasse, G. Paquin, X. Provençal '09]** Take any admissible Jacobi-Perron expansion. The boundaries of the patterns

\[
E_1^*(\sigma(B_1,C_1)) \cdots E_1^*(\sigma(B_n,C_n))(\mathcal{U})
\]

are self-avoiding paths.
Back to tilings
Long-range aperiodic order

Discrete planes with irrational normal vector are

- repetitive (*uniform recurrence*)
- aperiodic

The corresponding tilings are obtained by a *cut and project scheme* and yield *quasicrystals* (*model sets*)
Back to tilings  
Long-range aperiodic order

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The corresponding tilings are obtained by a cut and project scheme and yield quasicrystals (model sets)

Assume we have a "substitutive" arithmetic discrete plane

Multidimensional substitutive tilings \( \leadsto \) Local/matching rules [S. Mozes, C. Goodman-Strauss]

Can we recognize a "substitutive" arithmetic discrete plane by local inspection?...
.....up to a set of zero measure
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Multidimensional substitutive tilings \( \rightsquigarrow \) Local/matching rules [S. Mozes, C. Goodman-Strauss]

Can we recognize a "substitutive" arithmetic discrete plane by local inspection?...
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Yes in the Tribonacci case \( \sigma : 1 \mapsto 12, \ 2 \mapsto 13, \ 3 \mapsto 1 \) [X. Bressaud, M. Sablik, N. Pytheas Fogg ’09]
Towards new Interactions between Mathematics and Computer Science
February 01-March 05 2010
http://www.lirmm.fr/MathInfo2010/

• Lattice reduction
• Dynamics and Computation
• Multi-dimensional subshifts and tilings
• Sage days
• Topological Methods for the study of discrete structures