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Around the cap set conjecture

A very popular question in extremal combinatorics, inspired by Roth's theorem on sets of integers with no three in arithmetic progression, is the following:

Let S be a subset of $(\mathbf{Z}/3\mathbf{Z})^n$ such that no three distinct elements of S sum to 0. Such an S is called a *cap set*. How large can S be?

This simple problem has generated a rich stream of mathematics; it seems to sit at the intersection of many fields, from combinatorics to geometry to harmonic analysis. It was notable for the lack of consensus concerning its answer: the largest known cap sets were of size c^n for $c \sim 2.2$, while the best upper bounds on cap sets were of the form $n^{-1-\epsilon}3^n$. Is the true upper bound close to the latter, or more like C^n for some $C < 3$?

In 2016, work of Croot, Lev, Pach, myself, and Gijswijt showed the latter is the case, in a new application of the "polynomial method" from algebraic geometry. As is often the case with the polynomial method, the new proof is extremely short; so short I can, and will, explain it in an hour, and talk about some generalizations, too.