Cyclic decompositions of complete and complete multipartite hypergraphs

Shonda Gosselin, University of Winnipeg

Joint work with Artur Szymański and Adam Paweł Wojda, AGH University of Science and Technology, Kraków, Poland.

CanaDAM 2103
Memorial University of Newfoundland
June 12, 2013
Outline

1. Cyclic partitions
2. Generating $t$-complementary hypergraphs
3. Complete multipartite hypergraphs
Cyclic partitions

Definition

A cyclic t-partition of a hypergraph \((V, E)\) is a partition of the (hyper)edge set \(E\) of the form \(\{F, F^\theta, F^{\theta^2}, \ldots, F^{\theta^{t-1}}\}\), where \(\theta\) is a permutation of the vertex set \(V\).

A cyclic \(t\)-partition of a hypergraph \((V, E)\) is a decomposition of the hypergraph into \(t\) isomorphic hypergraphs which are permuted cyclically by a permutation \(\theta\) of \(V\).
Complete uniform hypergraphs

**Definition**

The **complete k-uniform hypergraph** with vertex set $V$ has edge set $\binom{V}{k}$, the set of all $k$-element subsets of $V$.

We assume that a hypergraph with $n$ vertices has vertex set $V_n = \{1, 2, \ldots, n\}$.

The complete $k$-uniform hypergraph on $n$ vertices is denoted by $K_n^{(k)}$. 
**t-complementary hypergraphs**

- A cyclic 2-partition of the complete graph $K_n^{(2)}$ contains a self-complementary graph and its complement.

- An isomorphism between a self-complementary graph and its complement is called a **complementing permutation**.

- Analogously, each of the $t$-$k$-uniform hypergraphs in a cyclic $t$-partition of $K_n^{(k)}$ is called **t-complementary**, and the associated permutation $\theta$ is called a **$(t, k)$-complementing permutation**.
The problem of determining whether a given uniform hypergraph is $t$-complementary has the same complexity as the graph isomorphism problem. (M.J. Colbourn and C.J. Colbourn, 1978)

Instead, we solve the problem of determining whether a given permutation is a $(t, k)$-complementing permutation for some $t$-complementary $k$-uniform hypergraph.

We characterize the cycle type of such permutations.
1. Cyclic partitions

2. Generating \( t \)-complementary hypergraphs

3. Complete multipartite hypergraphs
Construct a 2-complementary 3-hypergraph of order 6.

$$\theta = (1 \ 2)(3 \ 4 \ 5 \ 6)$$

Let $A_1 = \{x, y, z\} \subset \{1, 2, 3, 4, 5, 6\}$

$\{x, y, z\} \ \{x, y, z\}^\theta \ \{x, y, z\}^{\theta^2} \ \{x, y, z\}^{\theta^3}$
Construct a 2-complementary 3-hypergraph of order 6.

\[ \theta = (1 \ 2)(3 \ 4 \ 5 \ 6) \]

Let \( A_1 = \{x, y, z\} \subset \{1, 2, 3, 4, 5, 6\} \)

\[ \{x, y, z\} \ {x, y, z}^\theta \ {x, y, z}^{\theta^2} \ {x, y, z}^{\theta^3} \]
Construct a 2-complementary 3-hypergraph of order 6.

$$\theta = (1\ 2)(3\ 4\ 5\ 6)$$

Let $$A_1 = \{x, y, z\} \subset \{1, 2, 3, 4, 5, 6\}$$

$$\{x, y, z\}^\theta \quad \{x, y, z\}^\theta \quad \{x, y, z\}^\theta$$
Construct a 2-complementary 3-hypergraph of order 6.

\[ \theta = (1 2)(3 4 5 6) \]

Let \( A_1 = \{x, y, z\} \subset \{1, 2, 3, 4, 5, 6\} \)

\{x, y, z\} \  \{x, y, z\}^\theta \  \{x, y, z\}^{\theta^2} \  \{x, y, z\}^{\theta^3}
Construct a 2-complementary 3-hypergraph of order 6.

\[ \theta = (1 \ 2)(3 \ 4 \ 5 \ 6) \]

Let \( A_1 = \{x, y, z\} \subset \{1, 2, 3, 4, 5, 6\} \)

\[ \{x, y, z\} \ \{x, y, z\}^\theta \ \{x, y, z\}^{\theta^2} \ \{x, y, z\}^{\theta^3} \]
Construct a 2-complementary 3-hypergraph of order 6.

\[ \theta = (1 \ 2)(3 \ 4 \ 5 \ 6) \]

Let \( A_1 = \{x, y, z\} \subset \{1, 2, 3, 4, 5, 6\} \)

\[ \{x, y, z\} \quad \{x, y, z\}^\theta \quad \{x, y, z\}^{\theta^2} \quad \{x, y, z\}^{\theta^3} \]
Construct a 2-complementary 3-hypergraph of order 6.

$$\theta = (1, 2)(3, 4, 5, 6)$$

Let $A_1 = \{x, y, z\} \subseteq \{1, 2, 3, 4, 5, 6\}$

$\{x, y, z\} \{x, y, z\}^\theta \{x, y, z\}^{\theta^2} \{x, y, z\}^{\theta^3}$

Take a new 3-subset $A_2$ not in the sequence above and repeat...

$A_1 = \{1, 2, 3\} \{1, 2, 4\} \{1, 2, 5\} \{1, 2, 6\}$

$A_2 = \{1, 3, 4\} \{2, 4, 5\} \{1, 5, 6\} \{2, 3, 6\}$

$A_3 = \{1, 4, 5\} \{2, 5, 6\} \{1, 3, 6\} \{2, 3, 4\}$

$A_4 = \{3, 4, 5\} \{4, 5, 6\} \{3, 5, 6\} \{3, 4, 5\}$

$A_5 = \{1, 3, 5\} \{2, 4, 6\}$

$A_6 = \{1, 4, 6\} \{2, 3, 5\}$

Take either the red or the blue edges from each orbit of $\theta$ on the 3-subsets of $V = \{1, 2, 3, 4, 5, 6\}$. There are $2^6$ self-complementary 3-hypergraphs on $V$ with $(2, 3)$-complementing permutation $\theta$ (at most $2^5$ up to isomorphism).
Construct a 2-complementary 3-hypergraph of order 6.

\[ \theta = (1, 2)(3, 4, 5, 6) \]

Let \( A_1 = \{x, y, z\} \subset \{1, 2, 3, 4, 5, 6\} \)
\[ \{x, y, z\} \{x, y, z\}^{\theta} \{x, y, z\}^{\theta^2} \{x, y, z\}^{\theta^3} \]
Take a new 3-subset \( A_2 \) not in the sequence above and repeat...

\[ A_1 = \{1, 2, 3\} \{1, 2, 4\} \{1, 2, 5\} \{1, 2, 6\} \]
\[ A_2 = \{1, 3, 4\} \{2, 4, 5\} \{1, 5, 6\} \{2, 3, 6\} \]
\[ A_3 = \{1, 4, 5\} \{2, 5, 6\} \{1, 3, 6\} \{2, 3, 4\} \]
\[ A_4 = \{3, 4, 5\} \{4, 5, 6\} \{3, 5, 6\} \{3, 4, 5\} \]
\[ A_5 = \{1, 3, 5\} \{2, 4, 6\} \]
\[ A_6 = \{1, 4, 6\} \{2, 3, 5\} \]

Take either the red or the blue edges from each orbit of \( \theta \) on the 3-subsets of \( V = \{1, 2, 3, 4, 5, 6\} \). There are \( 2^6 \) self-complementary 3-hypergraphs on \( V \) with (2, 3)-complementing permutation \( \theta \) (at most \( 2^5 \) up to isomorphism).
Construct a 2-complementary 3-hypergraph of order 6.

\[ \theta = (1, 2)(3, 4, 5, 6) \]

Let \( A_1 = \{x, y, z\} \subset \{1, 2, 3, 4, 5, 6\} \)
\[ \{x, y, z\} \{x, y, z\}^\theta \{x, y, z\}^{\theta^2} \{x, y, z\}^{\theta^3} \]

Take a new 3-subset \( A_2 \) not in the sequence above and repeat...

\[ A_1 = \{1, 2, 3\} \ {1, 2, 4} \ {1, 2, 5} \ {1, 2, 6} \]
\[ A_2 = \{1, 3, 4\} \ {2, 4, 5} \ {1, 5, 6} \ {2, 3, 6} \]
\[ A_3 = \{1, 4, 5\} \ {2, 5, 6} \ {1, 3, 6} \ {2, 3, 4} \]
\[ A_4 = \{3, 4, 5\} \ {4, 5, 6} \ {3, 5, 6} \ {3, 4, 5} \]
\[ A_5 = \{1, 3, 5\} \ {2, 4, 6} \]
\[ A_6 = \{1, 4, 6\} \ {2, 3, 5} \]

Take either the red or the blue edges from each orbit of \( \theta \) on the 3-subsets of \( V = \{1, 2, 3, 4, 5, 6\} \). There are \( 2^6 \) self-complementary 3-hypergraphs on \( V \) with (2, 3)-complementing permutation \( \theta \) (at most \( 2^5 \) up to isomorphism).
Construct a 2-complementary 3-hypergraph of order 6.

θ = (1, 2)(3, 4, 5, 6)

Let $A_1 = \{x, y, z\} \subset \{1, 2, 3, 4, 5, 6\}$

$\{x, y, z\} \theta \{x, y, z\} \theta^2 \{x, y, z\} \theta^3$

Take a new 3-subset $A_2$ not in the sequence above and repeat...

$A_1 = \{1, 2, 3\} \ {1, 2, 4} \ {1, 2, 5} \ {1, 2, 6}$

$A_2 = \{1, 3, 4\} \ {2, 4, 5} \ {1, 5, 6} \ {2, 3, 6}$

$A_3 = \{1, 4, 5\} \ {2, 5, 6} \ {1, 3, 6} \ {2, 3, 4}$

$A_4 = \{3, 4, 5\} \ {4, 5, 6} \ {3, 5, 6} \ {3, 4, 5}$

$A_5 = \{1, 3, 5\} \ {2, 4, 6}$

$A_6 = \{1, 4, 6\} \ {2, 3, 5}$

Take either the red or the blue edges from each orbit of θ on the 3-subsets of $V = \{1, 2, 3, 4, 5, 6\}$. There are $2^6$ self-complementary 3-hypergraphs on $V$ with (2, 3)-complementing permutation θ (at most $2^5$ up to isomorphism).
Construct a 2-complementary 3-hypergraph of order 6.

\[ \theta = (1, 2)(3, 4, 5, 6) \]

Let \( A_1 = \{x, y, z\} \subset \{1, 2, 3, 4, 5, 6\} \)
\[ \{x, y, z\} \{x, y, z\}^\theta \{x, y, z\}^{\theta^2} \{x, y, z\}^{\theta^3} \]
Take a new 3-subset \( A_2 \) not in the sequence above and repeat...

\[ A_1 = \{1, 2, 3\} \{1, 2, 4\} \{1, 2, 5\} \{1, 2, 6\} \]
\[ A_2 = \{1, 3, 4\} \{2, 4, 5\} \{1, 5, 6\} \{2, 3, 6\} \]
\[ A_3 = \{1, 4, 5\} \{2, 5, 6\} \{1, 3, 6\} \{2, 3, 4\} \]
\[ A_4 = \{3, 4, 5\} \{4, 5, 6\} \{3, 5, 6\} \{3, 4, 5\} \]
\[ A_5 = \{1, 3, 5\} \{2, 4, 6\} \]
\[ A_6 = \{1, 4, 6\} \{2, 3, 5\} \]
Take either the red or the blue edges from each orbit of \( \theta \) on the 3-subsets of \( V = \{1, 2, 3, 4, 5, 6\} \). There are \( 2^6 \) self-complementary 3-hypergraphs on \( V \) with (2, 3)-complementing permutation \( \theta \) (at most \( 2^5 \) up to isomorphism).
Construct a 2-complementary 3-hypergraph of order 6.

\[ \theta = (1, 2)(3, 4, 5, 6) \]

Let \( A_1 = \{x, y, z\} \subset \{1, 2, 3, 4, 5, 6\} \)

\[ \{x, y, z\} \{x, y, z\}^{\theta} \{x, y, z\}^{\theta^2} \{x, y, z\}^{\theta^3} \]

Take a new 3-subset \( A_2 \) not in the sequence above and repeat...

\[ A_1 = \{1, 2, 3\} \{1, 2, 4\} \{1, 2, 5\} \{1, 2, 6\} \]
\[ A_2 = \{1, 3, 4\} \{2, 4, 5\} \{1, 5, 6\} \{2, 3, 6\} \]
\[ A_3 = \{1, 4, 5\} \{2, 5, 6\} \{1, 3, 6\} \{2, 3, 4\} \]
\[ A_4 = \{3, 4, 5\} \{4, 5, 6\} \{3, 5, 6\} \{3, 4, 5\} \]
\[ A_5 = \{1, 3, 5\} \{2, 4, 6\} \]
\[ A_6 = \{1, 4, 6\} \{2, 3, 5\} \]

Take either the red or the blue edges from each orbit of \( \theta \) on the 3-subsets of \( V = \{1, 2, 3, 4, 5, 6\} \). There are \( 2^6 \) self-complementary 3-hypergraphs on \( V \) with \((2, 3)\)-complementing permutation \( \theta \) (at most \( 2^5 \) up to isomorphism).
Construct a 2-complementary 3-hypergraph of order 6.

\[ \theta = (1, 2)(3, 4, 5, 6) \]

Let \( A_1 = \{x, y, z\} \subset \{1, 2, 3, 4, 5, 6\} \)
\( \{x, y, z\} \{x, y, z\}^\theta \{x, y, z\}^{\theta^2} \{x, y, z\}^{\theta^3} \)
Take a new 3-subset \( A_2 \) not in the sequence above and repeat...

\[ A_1 = \{1, 2, 3\} \{1, 2, 4\} \{1, 2, 5\} \{1, 2, 6\} \]
\[ A_2 = \{1, 3, 4\} \{2, 4, 5\} \{1, 5, 6\} \{2, 3, 6\} \]
\[ A_3 = \{1, 4, 5\} \{2, 5, 6\} \{1, 3, 6\} \{2, 3, 4\} \]
\[ A_4 = \{3, 4, 5\} \{4, 5, 6\} \{3, 5, 6\} \{3, 4, 5\} \]
\[ A_5 = \{1, 3, 5\} \{2, 4, 6\} \]
\[ A_6 = \{1, 4, 6\} \{2, 3, 5\} \]
Take either the red or the blue edges from each orbit of \( \theta \) on the 3-subsets of \( V = \{1, 2, 3, 4, 5, 6\} \). There are \( 2^6 \) self-complementary 3-hypergraphs on \( V \) with \((2, 3)\)-complementing permutation \( \theta \) (at most \( 2^5 \) up to isomorphism).
Construct a 2-complementary 3-hypergraph of order 6.

$$\theta = (1, 2)(3, 4, 5, 6)$$

Let $$A_1 = \{x, y, z\} \subset \{1, 2, 3, 4, 5, 6\}$$

$$\{x, y, z\} \{x, y, z\}^\theta \{x, y, z\}^\theta \{x, y, z\}^\theta$$

Take a new 3-subset $$A_2$$ not in the sequence above and repeat...

$$A_1 = \{1, 2, 3\} \{1, 2, 4\} \{1, 2, 5\} \{1, 2, 6\}$$
$$A_2 = \{1, 3, 4\} \{2, 4, 5\} \{1, 5, 6\} \{2, 3, 6\}$$
$$A_3 = \{1, 4, 5\} \{2, 5, 6\} \{1, 3, 6\} \{2, 3, 4\}$$
$$A_4 = \{3, 4, 5\} \{4, 5, 6\} \{3, 5, 6\} \{3, 4, 5\}$$
$$A_5 = \{1, 3, 5\} \{2, 4, 6\}$$
$$A_6 = \{1, 4, 6\} \{2, 3, 5\}$$

Take either the red or the blue edges from each orbit of $$\theta$$ on the 3-subsets of $$V = \{1, 2, 3, 4, 5, 6\}$$. There are $$2^6$$ self-complementary 3-hypergraphs on $$V$$ with $$(2, 3)$$-complementing permutation $$\theta$$ (at most $$2^5$$ up to isomorphism).
Construct a 2-complementary 3-hypergraph of order 6.

\[ \theta = (1, 2)(3, 4, 5, 6) \]

Let \( A_1 = \{x, y, z\} \subset \{1, 2, 3, 4, 5, 6\} \)
\( \{x, y, z\} \ {\{x, y, z\}^{\theta}} \ {\{x, y, z\}^{\theta^2}} \ {\{x, y, z\}^{\theta^3}} \)

Take a new 3-subset \( A_2 \) not in the sequence above and repeat...

\[ \begin{align*}
A_1 &= \{1, 2, 3\} \ {\{1, 2, 4\}} \ {\{1, 2, 5\}} \ {\{1, 2, 6\}} \\
A_2 &= \{1, 3, 4\} \ {\{2, 4, 5\}} \ {\{1, 5, 6\}} \ {\{2, 3, 6\}} \\
A_3 &= \{1, 4, 5\} \ {\{2, 5, 6\}} \ {\{1, 3, 6\}} \ {\{2, 3, 4\}} \\
A_4 &= \{3, 4, 5\} \ {\{4, 5, 6\}} \ {\{3, 5, 6\}} \ {\{3, 4, 5\}} \\
A_5 &= \{1, 3, 5\} \ {\{2, 4, 6\}} \\
A_6 &= \{1, 4, 6\} \ {\{2, 3, 5\}}
\end{align*} \]

Take either the red or the blue edges from each orbit of \( \theta \) on the 3-subsets of \( V = \{1, 2, 3, 4, 5, 6\} \). There are \( 2^6 \) self-complementary 3-hypergraphs on \( V \) with (2, 3)-complementing permutation \( \theta \) (at most \( 2^5 \) up to isomorphism).
Construct a 2-complementary 4-hypergraph of order 18.

\[ \theta = (1, 2, 3, 4, 5, 6)(7, 8, 9, 10, 11, 12)(13, 14, 15, 16, 17, 18) \]

\[ A_1 = \{1, 4, 7, 10\} \{2, 5, 8, 11\} \{3, 6, 9, 12\} \]

\[ \cdots \]

An orbit of \( \theta \) on the 4-subsets of \( \mathbb{Z}_{18} \) has odd length. This implies that \( \theta \) is not a (2,4)-complementing permutation of \( V_{18} \).
Construct a 4-complementary graph on $\mathbb{Z}_8$

$\theta = (0, 1, 2, 3, 4, 5, 6, 7)$
Color the orbits of $\theta$ on the 2-subsets of $\mathbb{Z}_8$:

$\mathcal{O}_1: 01 \ 12 \ 23 \ 34 \ 45 \ 56 \ 67 \ 07$
$\mathcal{O}_2: 02 \ 13 \ 24 \ 35 \ 46 \ 57 \ 06 \ 17$
$\mathcal{O}_3: 03 \ 14 \ 25 \ 36 \ 47 \ 05 \ 16 \ 27$
$\mathcal{O}_4: 04 \ 15 \ 26 \ 37$

Every orbit has length divisible by 4. Choose the edges of one color from each orbit.

There are $4^4$ different 4-complementary graphs on $\mathbb{Z}_8$ with $(4, 2)$-complementing permutation $\theta$ (at most $4^3$ up to isomorphism).
Eg. Choose the same color from each orbit ...
Eg. Choose a different color from every orbit ...

\( O_1 \): red, \( O_2 \): blue, \( O_3 \): green, \( O_4 \): violet.
A permutation \( \theta \) of \( V \) is a \((t, k)\)-complementing permutation
\[
\iff \quad \text{the sequence } A, A^\theta, A^{\theta^2}, A^{\theta^3}, \ldots \text{ has length divisible by } t, \\
\text{for all } k\text{-subsets } A \text{ of } V, \\
\iff \quad A^{\theta^j} \neq A, \\
\text{for all } k\text{-subsets } A \text{ of } V, \text{ for all integers } j \not\equiv 0 \pmod{t}.\]
Theorem

Let $n, k, p$ and $\alpha$ be positive integers such that $k < n$ and $p$ is prime. Let $\theta$ be a permutation on $V_n$. Then $\theta$ is a $(p^\alpha, k)$-complementing permutation if and only if there is an integer $\ell \geq 0$ such that the union of the orbits of $\theta$ with cardinality not divisible by $p^{\ell+\alpha}$ has cardinality less than $k \mod p^{\ell+1}$. 
Theorem

Let $n, k, p$ and $\alpha$ be positive integers such that $k < n$ and $p$ is prime. Suppose that $k = \sum_{i \geq 0} k_i p^i$ and $n = \sum_{i \geq 0} n_i p^i$, where $0 \leq k_i < p$ and $0 \leq n_i < p$ for $i \in \mathbb{N}$. Then the following three statements are equivalent.

1. There exists a cyclic $p^{\alpha}$-partition of $K_{n}^{(k)}$.
2. There exists $\ell \in \mathbb{N}$ such that $k_{\ell} \neq 0$ and $n \mod p^{\ell+\alpha} < k \mod p^{\ell+1}$.
3. There exist $r, \ell \in \mathbb{N}$ with $r \leq \ell$ such that $n_{r} < k_{r}$, $n_{i} = 0$ for $\ell < i < \ell + \alpha$ whenever $\alpha > 1$, and $n_{i} = k_{i}$ for $r < i \leq \ell$ whenever $r < \ell$. 

(Gosselin, Szymański, Wojda, 2010)
Theorem

Let n, k, p and α be positive integers such that k < n and p is prime. Suppose that $k = \sum_{i \geq 0} k_i p^i$ and $n = \sum_{i \geq 0} n_i p^i$, where $0 \leq k_i < p$ and $0 \leq n_i < p$ for $i \in \mathbb{N}$. Then the following three statements are equivalent.

1. There exists a cyclic $p^\alpha$-partition of $\mathcal{K}_n^{(k)}$.

2. There exists $\ell \in \mathbb{N}$ such that $k_\ell \neq 0$ and $n \mod p^{\ell+\alpha} < k \mod p^{\ell+1}$.

3. There exist $r, \ell \in \mathbb{N}$ with $r \leq \ell$ such that $n_r < k_r$, $n_i = 0$ for $\ell < i < \ell + \alpha$ whenever $\alpha > 1$, and $n_i = k_i$ for $r < i \leq \ell$ whenever $r < \ell$. 
Theorem

Let \( n, k, p \) and \( \alpha \) be positive integers such that \( k < n \) and \( p \) is prime. Suppose that \( k = \sum_{i \geq 0} k_i p^i \) and \( n = \sum_{i \geq 0} n_i p^i \), where \( 0 \leq k_i < p \) and \( 0 \leq n_i < p \) for \( i \in \mathbb{N} \). Then the following three statements are equivalent.

1. There exists a cyclic \( p^\alpha \)-partition of \( K_n^{(k)} \).

2. There exists \( \ell \in \mathbb{N} \) such that \( k_\ell \neq 0 \) and \( n \mod p^{\ell+\alpha} < k \mod p^{\ell+1} \).

3. There exist \( r, \ell \in \mathbb{N} \) with \( r \leq \ell \) such that \( n_r < k_r \), \( n_i = 0 \) for \( \ell < i < \ell + \alpha \) whenever \( \alpha > 1 \), and \( n_i = k_i \) for \( r < i \leq \ell \) whenever \( r < \ell \).
Theorem

Let $n, k, p$ and $\alpha$ be positive integers such that $k < n$ and $p$ is prime. Suppose that $k = \sum_{i \geq 0} k_i p^i$ and $n = \sum_{i \geq 0} n_i p^i$, where $0 \leq k_i < p$ and $0 \leq n_i < p$ for $i \in \mathbb{N}$. Then the following three statements are equivalent.

1. There exists a cyclic $p^\alpha$-partition of $\mathcal{K}_n^{(k)}$.
2. There exists $\ell \in \mathbb{N}$ such that $k_\ell \neq 0$ and $n \mod p^{\ell+\alpha} < k \mod p^{\ell+1}$.
3. There exist $r, \ell \in \mathbb{N}$ with $r \leq \ell$ such that $n_r < k_r$, $n_i = 0$ for $\ell < i < \ell + \alpha$ whenever $\alpha > 1$, and $n_i = k_i$ for $r < i \leq \ell$ whenever $r < \ell$. 
Condition 3 means that the base-$p$ representations of $n$ and $k$ have the following form:

<table>
<thead>
<tr>
<th></th>
<th>\ldots</th>
<th>$\ell + \alpha$</th>
<th>$\ell + \alpha - 1$</th>
<th>\ldots</th>
<th>$\ell + 1$</th>
<th>$\ell$</th>
<th>\ldots</th>
<th>$r + 1$</th>
<th>$r$</th>
<th>$r - 1$</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>\ldots</td>
<td>*</td>
<td>0</td>
<td>\ldots</td>
<td>0</td>
<td>$n_\ell$</td>
<td>\ldots</td>
<td>$n_{r+1}$</td>
<td>$n_r$</td>
<td>*</td>
<td>\ldots</td>
</tr>
<tr>
<td></td>
<td></td>
<td>\parallel</td>
<td>\ldots</td>
<td></td>
<td>\parallel</td>
<td>\parallel</td>
<td>\wedge</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k$</td>
<td>\ldots</td>
<td>*</td>
<td>*</td>
<td>\ldots</td>
<td>*</td>
<td>$k_\ell$</td>
<td>\ldots</td>
<td>$k_{r+1}$</td>
<td>$k_r$</td>
<td>*</td>
<td>\ldots</td>
</tr>
</tbody>
</table>
Lemma

If \( q_1 q_2 \cdots q_m \) is the prime power factorization of \( t \), then \( \theta \) is a \((t, k)\)-complementing permutation if and only if \( \theta \) is a \((q_i, k)\)-complementing permutation for all \( i \in \{1, 2, \ldots, m\} \).

Corollary

Let \( t, k \) and \( n \) be positive integers, \( k \leq n \), let \( q_1 q_2 \cdots q_m \) be the prime power factorization of \( t \), where \( q_i = p_i^{\alpha_i} \) for \( 1 \leq i \leq m \). If there exists a cyclic \( t \)-partition of \( K_n^{(k)} \), then for each \( i \in \{1, 2, \ldots, m\} \) there is \( \ell_i \in \mathbb{N} \) such that \( k \ell_i \neq 0 \) and

\[
\text{n mod } p_i^{\ell_i + \alpha_i} < k \text{ mod } p_i^{\ell_i + 1}
\]
Every permutation of $V_{89}$ with one orbit of cardinality 64 and one orbit of cardinality 25 is $(2, 40)$-complementing. 

$2 = 2^1$, $40 = 2^5 + 2^3$, and 
$89 \mod 2^{5+1} = 25 < 40 = 40 \mod 2^{5+1}$.

Every permutation of $V_{89}$ with one orbit of cardinality 8 and one orbit of cardinality 81 is $(9, 40)$-complementing.

$9 = 3^2$, $40 = 3^3 + 3^2 + 3^1 + 3^0$, and 
$89 \mod 3^{2+2} = 8 < 13 = 40 \mod 3^{2+1}$.

However, there is no $(18, 40)$-complementing permutation of $V_{89}$.

Hence the necessary condition of the previous corollary is not sufficient.
Complete nonuniform hypergraphs

Definition

For a nonempty subset $K$ of $V_{n-1}$, the complete $K$-hypergraph of order $n$ is \( \left( V_n, \bigcup_{k \in K} \binom{V_n}{k} \right) \) and denoted by $K_n^{(K)}$.

If \( \{ E, E^\theta, E^{\theta^2}, \ldots, E^{\theta^{t-1}} \} \) is a cyclic $t$-partition of $K_n^{(K)}$, then $\theta$ is called a \((t, K)\)-complementing permutation.
(Gosselin, Szymański, Wojda, 2010)

**Lemma**

Let $K \subseteq V_{n-1}$. A permutation $\theta$ of $V_n$ is a $(t, K)$-complementing permutation if and only if $\theta$ is a $(t, k)$-complementing permutation for all $k \in K$.

**Theorem**

Let $n, k, p$ and $\alpha$ be positive integers such that $p$ is prime and $k < n$. A permutation $\theta$ of $V_n$ is $(p^\alpha, V_k)$-complementing if and only if the cardinality of any orbit of $\theta$ is divisible by $p^{\alpha+\beta}$, where $\beta = \lfloor \log_p k \rfloor$.

**Corollary**

Let $n, k, p$ and $\alpha$ be positive integers such that $p$ is prime and $k < n$. There is a cyclic $p^\alpha$-partition of $K_n^{(V_k)}$ if and only if $p^{\alpha+\beta}$ divides $n$, where $\beta = \lfloor \log_p k \rfloor$. 
(Gosselin, Szymański, Wojda, 2010)

**Lemma**

Let $K \subseteq V_{n-1}$. A permutation $\theta$ of $V_n$ is a $(t, K)$-complementing permutation if and only if $\theta$ is a $(t, k)$-complementing permutation for all $k \in K$.

**Theorem**

Let $n, k, p$ and $\alpha$ be positive integers such that $p$ is prime and $k < n$. A permutation $\theta$ of $V_n$ is $(p^\alpha, V_k)$-complementing if and only if the cardinality of any orbit of $\theta$ is divisible by $p^{\alpha+\beta}$, where $\beta = \lfloor \log_p k \rfloor$.

**Corollary**

Let $n, k, p$ and $\alpha$ be positive integers such that $p$ is prime and $k < n$. There is a cyclic $p^\alpha$-partition of $K_n^{(V_k)}$ if and only if $p^{\alpha+\beta}$ divides $n$, where $\beta = \lfloor \log_p k \rfloor$. 
Lemma

Let $K \subseteq V_{n-1}$. A permutation $\theta$ of $V_n$ is a $(t, K)$-complementing permutation if and only if $\theta$ is a $(t, k)$-complementing permutation for all $k \in K$.

Theorem

Let $n, k, p$ and $\alpha$ be positive integers such that $p$ is prime and $k < n$. A permutation $\theta$ of $V_n$ is $(p^\alpha, V_k)$-complementing if and only if the cardinality of any orbit of $\theta$ is divisible by $p^{\alpha + \beta}$, where $\beta = \lfloor \log_p k \rfloor$.

Corollary

Let $n, k, p$ and $\alpha$ be positive integers such that $p$ is prime and $k < n$. There is a cyclic $p^\alpha$-partition of $K_n^{(V_k)}$ if and only if $p^{\alpha + \beta}$ divides $n$, where $\beta = \lfloor \log_p k \rfloor$. 
Let $n, k$ and $q$ be positive integers such that $t = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_s^{\alpha_s}$, where $p_1, p_2, \ldots, p_s$ are mutually distinct primes, $k < n$, and $\beta_j = \lfloor \log_{p_j} k \rfloor$ for every $j = 1, 2, \ldots, s$. A permutation $\theta$ of $V_n$ is $(t, V_k)$-complementing if and only if the cardinality of any orbit of $\theta$ is divisible by $p_1^{\alpha_1+\beta_1} \cdot p_2^{\alpha_2+\beta_2} \cdot \ldots \cdot p_s^{\alpha_s+\beta_s}$. 
Outline

1. Cyclic partitions
2. Generating $t$-complementary hypergraphs
3. Complete multipartite hypergraphs
Complete multipartite hypergraphs

**Definition**

The **complete t-partite k-uniform hypergraph** with vertex set $V = A_1 \cup A_2 \cup \cdots \cup A_t$ has edge set $E_k = \{e : e \subseteq V, |e| = k, e \not\subseteq A_i \text{ for } i = 1, 2, \ldots, t\}$, and is denoted by $\mathcal{K}^{(k)}(A_1, A_2, \ldots, A_t)$ or $\mathcal{K}^{(k)}_{n_1, n_2, \ldots, n_t}$ when $|A_i| = n_i$ for $i = 1, 2, \ldots, t$.

**Definition**

For a nonempty set $K$ of positive integers, let $\mathcal{K}^{(K)}(A_1, A_2, \ldots, A_t)$ denote the hypergraph with vertex set $V = A_1 \cup A_2 \cup \ldots \cup A_t$ and edge set $\bigcup_{k \in K} E_k$. 

Cyclic partitions Generating $t$-complementary hypergraphs Complete multipartite hypergraphs
Complete multipartite hypergraphs

Definition
The complete $t$-partite $k$-uniform hypergraph with vertex set $V = A_1 \cup A_2 \cup \cdots \cup A_t$ has edge set $E_k = \{ e : e \subset V, |e| = k, e \not\subseteq A_i \text{ for } i = 1, 2, \ldots, t \}$, and is denoted by $\mathcal{K}^{(k)}(A_1, A_2, \ldots, A_t)$ or $\mathcal{K}_{n_1, n_2, \ldots, n_t}^{(k)}$ when $|A_i| = n_i$ for $i = 1, 2, \ldots, t$.

Definition
For a nonempty set $K$ of positive integers, let $\mathcal{K}^{(K)}(A_1, A_2, \ldots, A_t)$ denote the hypergraph with vertex set $V = A_1 \cup A_2 \cup \cdots \cup A_t$ and edge set $\bigcup_{k \in K} E_k$. 
Cyclic partitions of complete multipartite hypergraphs

Let $k, p, t$ and $\alpha$ be positive integers such that $p$ is prime. Let $V = A_1 \cup A_2 \cup ... \cup A_t$ where $A_i \cap A_j = \emptyset$ for $i \neq j$.

Let $\theta$ be a permutation of the set $V$ such that $A_i^\theta = A_i$ for every $i = 1, 2, ..., t$.

When does $\theta$ induce a cyclic $p^\alpha$-partition of $\mathcal{K}(V_k)(A_1, A_2, ..., A_t)$?

When the orbit of $\theta$ on any $e \in E_r$ has length divisible by $p^\alpha$, for all $r \in V_k$.

This is guaranteed if $\theta|_{A_i}$ is a $(p^\alpha, V_{k-1})$-complementing permutation for all but at most one $i$,

or if $\theta|_{A_i}$ is a $(p^\alpha, V_{k/2})$-complementing permutation for all $i$. 
Cyclic partitions of complete multipartite hypergraphs

Let $k, p, t$ and $\alpha$ be positive integers such that $p$ is prime. Let $V = A_1 \cup A_2 \cup \ldots \cup A_t$ where $A_i \cap A_j = \emptyset$ for $i \neq j$.

Let $\theta$ be a permutation of the set $V$ such that $A_i \theta = A_i$ for every $i = 1, 2, \ldots, t$.

When does $\theta$ induce a cyclic $p^\alpha$-partition of $\mathcal{K}(V_k)(A_1, A_2, \ldots, A_t)$?

When the orbit of $\theta$ on any $e \in E_r$ has length divisible by $p^\alpha$, for all $r \in V_k$.

This is guaranteed if $\theta|_{A_i}$ is a $(p^\alpha, V_{k-1})$-complementing permutation for all but at most one $i$,

or if $\theta|_{A_i}$ is a $(p^\alpha, V_{k/2})$-complementing permutation for all $i$. 
Cyclic partitions of complete multipartite hypergraphs

Let \( k, p, t \) and \( \alpha \) be positive integers such that \( p \) is prime. Let \( V = A_1 \cup A_2 \cup \ldots \cup A_t \) where \( A_i \cap A_j = \emptyset \) for \( i \neq j \).

Let \( \theta \) be a permutation of the set \( V \) such that \( A_i^\theta = A_i \) for every \( i = 1, 2, \ldots, t \).

When does \( \theta \) induce a cyclic \( p^\alpha \)-partition of \( K^{(V_k)}(A_1, A_2, \ldots, A_t) \)?

When the orbit of \( \theta \) on any \( e \in E_r \) has length divisible by \( p^\alpha \), for all \( r \in V_k \).

This is guaranteed if \( \theta|_{A_i} \) is a \( (p^\alpha, V_{k-1}) \)-complementing permutation for all but at most one \( i \),

or if \( \theta|_{A_i} \) is a \( (p^\alpha, V_{k/2}) \)-complementing permutation for all \( i \).
Let $k, p, t$ and $\alpha$ be positive integers such that $p$ is prime. Let $V = A_1 \cup A_2 \cup \ldots \cup A_t$ where $A_i \cap A_j = \emptyset$ for $i \neq j$.

Let $\theta$ be a permutation of the set $V$ such that $A_i^\theta = A_i$ for every $i = 1, 2, \ldots, t$.

When does $\theta$ induce a cyclic $p^\alpha$-partition of $\mathcal{K}(V_k)(A_1, A_2, \ldots, A_t)$?

When the orbit of $\theta$ on any $e \in E_r$ has length divisible by $p^\alpha$, for all $r \in V_k$.

This is guaranteed if $\theta|_{A_i}$ is a $(p^\alpha, V_{k-1})$-complementing permutation for all but at most one $i$,
or if $\theta|_{A_i}$ is a $(p^\alpha, V_{k/2})$-complementing permutation for all $i$. 
Let \( k, p, t \) and \( \alpha \) be positive integers such that \( p \) is prime. Let \( V = A_1 \cup A_2 \cup \ldots \cup A_t \) where \( A_i \cap A_j = \emptyset \) for \( i \neq j \).

Let \( \theta \) be a permutation of the set \( V \) such that \( A_i^{\theta} = A_i \) for every \( i = 1, 2, \ldots, t \).

When does \( \theta \) induce a cyclic \( p^\alpha \)-partition of \( K(V_k)(A_1, A_2, \ldots, A_t) \)?

When the orbit of \( \theta \) on any \( e \in E_r \) has length divisible by \( p^\alpha \), for all \( r \in V_k \).

This is guaranteed if \( \theta|_{A_i} \) is a \( (p^\alpha, V_k-1) \)-complementing permutation for all but at most one \( i \),

or if \( \theta|_{A_i} \) is a \( (p^\alpha, V_k/2) \)-complementing permutation for all \( i \).
Cyclic partitions of complete multipartite hypergraphs

Let \( k, p, t \) and \( \alpha \) be positive integers such that \( p \) is prime. Let \( V = A_1 \cup A_2 \cup \ldots \cup A_t \) where \( A_i \cap A_j = \emptyset \) for \( i \neq j \).

Let \( \theta \) be a permutation of the set \( V \) such that \( A_i^\theta = A_i \) for every \( i = 1, 2, \ldots, t \).

When does \( \theta \) induce a cyclic \( p^\alpha \)-partition of \( \mathcal{K}^{(V_k)}(A_1, A_2, \ldots, A_t) \)?

When the orbit of \( \theta \) on any \( e \in E_r \) has length divisible by \( p^\alpha \), for all \( r \in V_k \).

This is guaranteed if \( \theta|_{A_i} \) is a \( (p^\alpha, V_{k-1}) \)-complementing permutation for all but at most one \( i \),

or if \( \theta|_{A_i} \) is a \( (p^\alpha, V_{k/2}) \)-complementing permutation for all \( i \).
Theorem

... Each of the following two conditions is sufficient for \( \theta \) to induce a cyclic \( p^\alpha \)-partition of \( \mathcal{K}(V_k)(A_1, A_2, ..., A_t) \):

1. For all but at most one \( i \in \{1, 2, ..., t\} \), the cardinality of all the orbits of \( \theta|_{A_i} \) are divisible by \( p^{\alpha+\beta} \), where \( \beta = \lfloor \log_p (k - 1) \rfloor \).

2. For every \( i \in \{1, 2, ..., t\} \), the cardinalities of all orbits of \( \theta|_{A_i} \) are divisible by \( p^{\alpha+\gamma} \), where \( \gamma = \lfloor \log_p k/2 \rfloor \).
Corollary

Let $n_1, n_2, \ldots, n_t, k, p$ and $\alpha$ be positive integers such that $p$ is prime. If at least one of the following two conditions is verified then there is a cyclic $p^\alpha$-partition of $\mathcal{K}(V_k)^{(V_k)}_{n_1, n_2, \ldots, n_t}$.

1. For all but at most one $i \in \{1, 2, \ldots, t\}$, $p^{\alpha + \beta} | n_i$ where $\beta = \lfloor \log_p (k - 1) \rfloor$.
2. For every $i \in \{1, 2, \ldots, t\}$, $p^{\alpha + \gamma} | n_i$ where $\gamma = \lfloor \log_p k/2 \rfloor$. 
Thank You!