Algebraic Hypergraph Decompositions

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Paley graphs  
t-complementary hypergraphs  
Generalized Paley hypergraphs
Outline

Paley graphs

t-complementary hypergraphs

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$t$-complementary hypergraphs

Generalized Paley hypergraphs
The Paley graph

Definition
For a prime power \( q \equiv 1 \pmod{4} \) and a finite field \( \mathbb{F}_q \), the **Paley graph of order** \( q \), denoted by \( \text{Paley}(q) \), is the simple graph with vertex set \( V = \mathbb{F}_q \) and edge set \( E \), where

\[ \{x, y\} \in E \iff x - y \text{ is a nonzero square.} \]
Paley graphs

$t$-complementary hypergraphs

Generalized Paley hypergraphs

Paley(13)
Paley graphs

**Paley(13)**
Paley graphs

- $5$-Paley graph
- $9$-Paley graph
- $13$-Paley graph
- $17$-Paley graph
- $25$-Paley graph
- $29$-Paley graph
- $37$-Paley graph
- $41$-Paley graph
Paley graphs

**Paley**$(q)$ **is self-complementary**

If $\omega$ is a generator of $\mathbb{F}_q^*$, then

$$x - y \in \langle \omega^2 \rangle \iff \omega x - \omega y = \omega(x - y) \notin \langle \omega^2 \rangle.$$ 

$T_{\omega,0} : x \mapsto \omega x$ is an isomorphism from $Paley(q)$ to its complement. □
**Paley**($q$) is vertex-transitive

For $b \in \mathbb{F}_q$,

$$x - y \in \langle \omega^2 \rangle \iff (x + b) - (y + b) = x - y \in \langle \omega^2 \rangle$$

$T_{1,b} : x \mapsto x + b$ is an automorphism of $\text{Paley}(q)$.  

$\{ T_{1,b} : b \in \mathbb{F}_q \}$ acts transitively on $\mathbb{F}_q$.  

$\text{Aut}(\text{Paley}(q))$ is an index-2 subgroup of the affine group $A\Gamma L(1, q)$
Outline

Paley graphs

$t$-complementary hypergraphs

Generalized Paley hypergraphs
Definition
A simple $k$-uniform hypergraph $X$ with vertex set $V$ and edge set $E$ is **$t$-complementary** if there is a permutation $\theta$ on $V$ such that the sets

$$E, E^\theta, E^{\theta^2}, \ldots, E^{\theta^{t-1}}$$

partition the set of $k$-subsets of $V$.

$\theta$ is called a **$t$-antimorphism** of $X$ (i.e., $\theta \in \text{Ant}_t(X)$).
• The 2-complementary 2-uniform hypergraphs are the **self-complementary graphs**, which have been well studied due to their connection to the graph isomorphism problem.

• The $t$-complementary $k$-hypergraphs correspond to **cyclic edge decompositions (cyclotomic factorisations)** of the complete $k$-uniform hypergraph into $t$ parts.

• The vertex-transitive $t$-complementary $k$-uniform hypergraphs correspond to **large sets of isomorphic designs** which are point-transitive.
Outline

Paley graphs

t-complementary hypergraphs

Generalized Paley hypergraphs
The Paley graph - revisited

Definition
For a prime power $q \equiv 1 \pmod{4}$ and a finite field $\mathbb{F}_q$ of order $q$, the **Paley graph of order $q$**, denoted by $\text{Paley}(q) = (V, E)$, is the simple graph with $V = \mathbb{F}_q$ and

$$\{x, y\} \in E \iff x - y \in \langle \omega^2 \rangle$$

where $\omega$ is a generator of $\mathbb{F}_q^*$. 
Constructing $t$-complementary $k$-hypergraphs

Partition a group $G$ into $t$ sets

$$C_0, C_1, \ldots, C_{t-1},$$

where each $C_i$ is a union of cosets of a subgroup $S$ of $G$.

Find an operation $\Psi : \binom{V}{k} \to G$ and a permutation $\theta : V \to V$ such that

$$\Psi(\{x_1, \ldots, x_k\}) \in C_i \iff \Psi(\{x_1, \ldots, x_k\}^\theta) \in C_{i+s}$$

for some $s$ where $\gcd(s, t) = 1$.

Let $E_i = \left\{ e \in \binom{V}{k} : \Psi(e) \in C_i \right\}$.

Then $X_i = (V, E_i)$ is $t$-complementary with $t$-antimorphism $\theta$. 
Examples

1. Paley Graphs:
   - \( V = \mathbb{F}_q \).
   - \( G = \mathbb{F}_q^* \).
   - \( S = \langle \omega^2 \rangle \).
   - \( \Psi(\{x, y\}) = x - y \).

2. Generalized Paley \( k \)-hypergraphs:
   - \( V = \mathbb{F}_q \).
   - \( G \) is the group of squares of \( \mathbb{F}_q^* \).
   - \( S = \langle \omega^{2t(\binom{k}{2})} \rangle \).
   - \( \Psi(\{x_1, x_2, \ldots, x_k\}) = \prod_{i<j}(x_i - x_j)^2 \).
The Generalized Paley Hypergraph  \( \text{Paley}(q, k, t) \)

**Definition**

- \( t \) is prime, \( \ell \) is the highest power of \( t \) dividing \( k \) or \( k - 1 \).
- \( q \) is a prime power, \( q \equiv 1 \pmod{t^{\ell+1}} \)
- \( G \) is the group of squares in \( \mathbb{F}_q^* \).
- \( S = \langle \omega^{2t} \binom{k}{2} \rangle \).
- \( c = \gcd(|G|, \binom{k}{2}) \).  \( t \)\(c \) is the number of cosets of \( S \) in \( G \).
- \( F_i \) is the coset \( \omega^{2i} \langle \omega^{2t} \binom{k}{2} \rangle \) in \( G \) \((0 \leq i \leq tc - 1)\).
- \( C_j = F_{jc+0} \cup F_{jc+1} \cup \cdots \cup F_{(j+1)c-1} \) \((0 \leq j \leq t - 1)\).

The **Generalized Paley Hypergraph**  \( \text{Paley}(q, k, t) = (V, E) \) is the simple \( k \)-hypergraph with \( V = \mathbb{F}_q \) and

\[
\{x_1, x_2, \ldots, x_k\} \in E \iff \prod_{i<j} (x_i - x_j)^2 \in C_0.
\]
$P(q, k, t)$ is $t$-complementary

\[
\prod_{i < j} (x_i - x_j)^2 \in F_i
\]

\[
\iff \prod_{i < j} (\omega x_i - \omega x_j)^2 = \omega^{2(\frac{k}{2})} \prod_{i < j} (x_i - x_j)^2 \in F_{i+sc},
\]

where $\gcd(s, t) = 1$.

$T_{\omega, 0} : x \rightarrow \omega x$ is a $t$-antimorphism of $Paley(q, k, t)$. 
$Paley(q, k, t)$ is vertex-transitive

For $b \in \mathbb{F}_q$,

$$\prod_{i<j}(x_i - x_j)^2 \in F_i$$

$$\iff \prod_{i<j}((x_i + b) - (x_j + b))^2 = \prod_{i<j}(x_i - x_j)^2 \in F_i.$$

$T_{1,b} : x \rightarrow x + b$ is an automorphism of $Paley(q, k, t)$.
Automorphisms and $t$-antimorphisms of $Paley(q, k, t)$

\[ \text{Aut}(Paley(q, k, t)) \geq \{ T_{a,b} \mid a = \omega^s, s \equiv 0 \pmod{t}, b \in \mathbb{F}_q \} \]

\[ \text{Ant}_t(Paley(q, k, t)) \supseteq \{ T_{a,b} \mid a = \omega^s, s \not\equiv 0 \pmod{t}, b \in \mathbb{F}_q \}. \]

\[ T_{a,b} : x \mapsto ax + b \]

$\text{Aut}(Paley(q, k, t))$ contains an index-$t$ subgroup of the affine group $\text{AGL}(1, q)$. 
Generalized Paley hypergraph constructions

- $t = 2, k = 2$ (Paley)
- $t = 2, k = 3$ (Kocay, 1992)
- $t = 2, k = 2$, $r$-factor (Peisert, 2001)
- $t, k = 2$ (Li, Praeger 2003)(Li, Lim and Praeger 2009)
- $t = 2$, any $k$ (Potočnik and Šajna, 2009)
- Odd prime $t$, any $k$, $r$-factor (G. 2011)
$n$ not a prime power?
Construction: Generalized Paley $k$-hypergraph

- $n \geq k$, $n = q_1 q_2 \cdots q_s$ is the prime power decomposition of $n$.

- $\ell$ is the largest power of $t$ that divides $m$ or $m - 1$ for $2 \leq m \leq k$.

- $q_i \equiv 1 \pmod{t^{\ell + 1}}$ for $i = 1, 2, \ldots, s$.

- $V := F_{q_1} \times F_{q_2} \times \cdots \times F_{q_s}$.

- Define $Paley(n, k, t) = (V, E)$, where ....
\[
E = \left\{ \{x_{11}, x_{12}, \ldots, x_{1j}, \ldots, x_{1s}\}, \{x_{21}, x_{22}, \ldots, x_{2j}, \ldots, x_{2s}\}, \{x_{31}, x_{32}, \ldots, x_{3j}, \ldots, x_{3s}\}, \ldots, \{x_{kj}, x_{kj}, \ldots, x_{ks}\} \right\}
\]

\(j\) is the smallest integer in \(\{1, 2, \ldots, s\}\) for which \(j\)-th coordinates of the elements in \(E\) are not all equal.

\(E \in \mathcal{E}\) if and only if the \(j\)-th coordinates form an edge of \(\text{Paley}(q_j, k, t)\).

\(\text{Paley}(n, k, t)\) is vertex-transitive and \(t\)-complementary.
Conditions on order

Theorem
Let $t$ be prime, let $\ell$ and $b$ be positive integers such that $1 \leq b \leq t - 1$, and suppose that $k$ or $k - 1$ equals $bt^\ell$. Suppose $n$ is a positive integer, $n > k$, and $n = q_1 q_2 \cdots q_s$ is its prime power decomposition. If $n \equiv 1 \pmod{t^{\ell+1}}$, then there exists a vertex-transitive $t$-complementary $k$-hypergraph of order $n$ if and only if

$q_i \equiv 1 \pmod{t^{\ell+1}} \quad \text{for } 1 \leq i \leq s.$

Necessity: $t = 2, k = 2$ (Muzychuk, 1992);
$t = 2$ (Potočnik, Šajna, 2007); $t$ prime (G. 2010)

Sufficiency: $t = 2, k = 2$ (Rao, 1985);
$t = 2$, any $k$, $n$ a prime power (Potočnik, Šajna, 2009);
$t$ prime, any $k$, any $n$ (G. 2010)
Conditions on order

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$$q_i \equiv 1 \pmod{t^{\ell+1}} \quad \text{for} \ 1 \leq i \leq s.$$  

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Raymond Paley (1907-1933)