Equality in the Domination Chain in Triangulations

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Joint work with C. M. van Bommel

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1. Definitions and Introduction

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4. Triangulation
Independence and Domination

- An *independent set* is a set where no two vertices are adjacent.
- An *dominating set* is a set who is adjacent to every vertex in the graph.
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![Graph Diagram]

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  - What property makes a dominating set minimal?
    - Private Neighbours

![Graph Diagram]

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Independence and Domination

- An *independent set* is a set where no two vertices are adjacent.
- An *dominating set* is a set who is adjacent to every vertex in the graph.

What property makes a dominating set minimal?

- Private Neighbours: A private neighbour of \(x\) in the set \(I\) are the vertices in \(N[x] - N[I - \{x\}]\).
Irredundance

- A dominating set is minimal provided every vertex in the set has a private neighbour.
- An *irredundant set* is a set where every vertex has a private neighbour.
- Private Neighbours: A private neighbour of $x$ in the set $I$ are the vertices in $N[x] - N[I - \{x\}]$. 

Diagram:

```
  a -- b -- a
   \  |  /   \
    c- d -- c
  /  \   /  \   |
 e- x- f- e- x- f
/  \   /  \   /  \
 g- h- g- h- h
/  \   /  \   |
 i- j- i- j- j
```
A dominating set is minimal provided every vertex in the set has a private neighbour.

An *irredundant set* is a set where every vertex has a private neighbour.

Private Neighbours: A private neighbour of $x$ in the set $I$ are the vertices in $N[x] - N[I - \{x\}]$. 
For a graph $G$:

- $i(G)$, is the minimum cardinality of a maximal independent set of $G$
- $\alpha(G)$, is the maximum cardinality of an independent set of $G$
- $\gamma(G)$, is the minimum cardinality of a dominating set of $G$
- $\Gamma(G)$, is the maximum cardinality of a minimal dominating sets of $G$
- $ir(G)$, is the minimum cardinality of a maximal irredundant set of $G$
- $IR(G)$, is the maximum cardinality of an irredundant set of $G$
Every maximal independent set is a minimal dominating set and

Every minimal dominating set is a maximal irredundant set

This implies a relation of inequalities between the parameters, widely known as the *domination chain*:

\[
ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G)
\]
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\[ \text{ir}(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G) \]

\[ \text{ir}(G) = 4 \quad \gamma(G) = i(G) = 5 \]
\[ \alpha(G) = \Gamma(G) = IR(G) = 6 \]
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\[ \text{ir}(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G) \]

- When can equality hold in the various parts of the domination chain?
- When are all six domination parameters equal?
When can equality hold in the various parts of the domination chain?

When are all six domination parameters equal?
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Well-covered Graphs

\[ i(G) = \alpha(G) \]
Well-covered Graphs

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i(G) = \alpha(G)

- The general recognition problem is co-NP-complete [Sankaranarayana, Stewart, 1992 and Chvátal, Slater, 1993]
- Polynomial for:
  - Girth 5 graphs [A. Finbow, Hartnell, Nowakowski, 1993]
  - No 4 and 5 cycles [A. Finbow, Hartnell, Nowakowski, 1994]
  - Claw-free graphs [Tankus, Tarsi, 1996]
  - Chordal graphs [Prisner, Topp, Vestergaard, 1996]
  - Graphs of bounded degree [Caro, Ellingham, Ramey, 1998]
  - Planar, 3-connected cubic Claw-free graphs [King, 2003]
  - No 3, 5 nor 7 cycles [Randerath, Vestergaard, 2006]
  - Planar, 4 connected triangulations [A. Finbow, Hartnell, Nowakowski, Plummer 2004-2013+]
Well-covered Graphs

\[ i(G) = \alpha(G) \]

- Open Questions:
  - Planar graphs
  - Graphs with no 4-cycles
  - Cartesian Products
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Equality in the Domination Chain

Questions

\[ ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G) \]

- When can equality hold in the various parts of the domination chain?
- When are all six domination parameters equal?
- For which graphs do all minimal dominating sets have the same cardinality?
- Complexity issues are not settled.
When can equality hold in the various parts of the domination chain?

When are all six domination parameters equal?

For which graphs do all minimal dominating sets have the same cardinality?

Complexity issues are not settled.

We focus on planar triangulations.
1. Definitions and Introduction

2. Well-covered Graphs

3. Equality in the Domination Chain

4. Triangulation
A triangulation is a maximal planar graph.
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Well-covered Triangulations

Let $G$ be a planar triangulation. Then $G$ is well-covered if and only if:
Let $G$ be a planar triangulation. Then $G$ is well-covered if and only if $G$ is a member of the $K_4$-family.
Well-covered Triangulations

Let $G$ be a planar triangulation. Then $G$ is well-covered if and only if: $G$ is a member of the $K_4$-family.
OR
Well-covered Triangulations

Let $G$ be a planar triangulation. Then $G$ is well-covered if and only if: $G$ is a member of the extended $K_4$-family.
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Let $G$ be a planar triangulation. Then $G$ is well-covered if and only if $G$ is a member of the extended $K_4$-family or else $G$ is one of the following graphs: $K_3$, $R_6$, $R_7$, $R_8$, $R_{12}$, $R_8 \bigcirc K_3$, or $R_8 \bigcirc R_8$. 
Let $G$ be a member of the $K_4$-family with

$$ir(G) = \gamma(G) = i(G) = \alpha(G) = \Gamma(G) = IR(G)$$
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Equality in the domination chain

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Bad Configurations
Good Configurations

[Diagram of two triangulations with labeled vertices u1, u2, u3, u4, v1, v2, v3, v4.]

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Equality in the Domination Chain in Triangulations
Let $G$ be a planar triangulation. Then $ir(G) = IR(G)$ if and only if either:

1. $G$ is one of the following graphs: $K_4 \circ K_3$, $K_4 \circ R_8$, $K_3$, $R_6$, $R_7$, $R_8$, $R_{12}$, or $R_8 \circ K_3$. 

![Graphs](image.png)
Equality in the domination chain

Let $G$ be a planar triangulation. Then $ir(G) = IR(G)$ if and only if either:

1. $G$ is in the $K_4$ family and for each copy $U$ of $K_4$, one of the following conditions hold:
   1. There exists an ordering of the exterior vertices such that $N[u_4] \subseteq N[u_1] \subseteq N[u_2] \subseteq N[u_3]$ in $G$. 

![Diagram of a planar triangulation with exterior vertices ordered as described in the text.](attachment:image.png)
OR
Equality in the domination chain

Let $G$ be a planar triangulation. Then $ir(G) = IR(G)$ if and only if either:

1. $G$ is in the $K_4$ family and for each copy $U$ of $K_4$, one of the following conditions hold:
   2. There exists a copy $V$ of $K_4$ such that:
      - There exists an ordering of the exterior vertices such that $N[u_4] \subseteq N[u_1] \subseteq N[u_2] \subseteq N[u_3]$ in $G - V$.
      - At most one cycle of is not a face of $G$.
      - If $u_1u_3v_1$ is not a face of $G$, then each neighbour of $u_1$ or $u_3$ in $G - U - V$ is adjacent to each neighbour of $v_1$ in $G - U - V$.  

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Equality in the Domination Chain in Triangulations
OR
Equality in the domination chain

Let $G$ be a planar triangulation. Then $ir(G) = IR(G)$ if and only if either:

- (2) $G$ is in the $K_4$ family and for each copy $U$ of $K_4$, one of the following conditions hold:
  - (c) There exists a copy $V$ of $K_4$ such that:
    - There exists an ordering of the exterior vertices such that $N[u_4] \subseteq N[u_1] \subseteq N[u_2] \subseteq N[u_3]$ in $G - V$.
    - For each neighbour $z_i$ of $u_1$, either $z_i \sim u_2$, or $z_i$ is adjacent to each neighbour of $v_1$ that is not also adjacent to $v_2$. 

![Diagram showing the conditions for equality in the domination chain in planar triangulations](image_url)
OR
Let $G$ be a planar triangulation. Then $ir(G) = IR(G)$ if and only if either:

1. $G$ is in the $K_4$ family and $G$ is:

Let $G$ be a planar triangulation. Then $ir(G) = IR(G)$ if and only if either:

1. $G$ is in the $K_4$ family and $G$ is:
The End

Thank you!!!

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