The asymptotic value of the independence ratio for the direct graph power

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**Independence ratio** of a graph $G$: $i(G) = \frac{\alpha(G)}{|V(G)|}$

**Direct product** of two graphs $G$ and $H$: the graph $G \times H$ for which

$V(G \times H) = V(G) \times V(H)$, and

$\{(x_1, y_1), (x_2, y_2)\} \in E(G \times H)$, iff

$\{x_1, x_2\} \in E(G)$ and $\{y_1, y_2\} \in E(H)$.

$G^{\times k}$ denotes the $k$th direct power of $G$
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**Definition** (Brown, Nowakowski, Rall - 1996.):
The asymptotic value of the independence ratio for the direct graph power is defined as

$$A(G) = \lim_{k \to \infty} i(G \times^k).$$
Results of Brown, Nowakowski and Rall

\[0 < i(G) \leq i(G \times 2) \leq i(G \times 3) \leq \cdots \leq A(G) \leq 1\]

**Theorem** (Brown, Nowakowski, Rall - 1996.):
For any independent set \(U\) of \(G\) we have \(A(G) \geq \frac{|U|}{|U| + |N_G(U)|}\), where \(N_G(U)\) denotes the neighbourhood of \(U\) in \(G\).
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there exists an independent set \( U_k \) of \( G^{\times k} \) such that

\[ \frac{|U_k|}{|U_k| + |N_{G^{\times k}}(U_k)|} \geq \frac{|U|}{|U| + |N_G(U)|} \]

and

\[ \lim_{k \to \infty} \frac{|U_k|}{|V(G^{\times k})|} = \frac{|U|}{|U| + |N_G(U)|} \]
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if \( \alpha(G) = \frac{1}{2}|V(G)| \) then

\( G \) has a perfect matching,

therefore \( G^{\times k} \) also has one (\( \forall k \))

and \( i(G^{\times k}) \leq \frac{1}{2} \) thus \( A(G) = \frac{1}{2} \)
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**Observation** (Alon, Lubetzky): \( A(G) \geq i^*_\text{max}(G) \), where

\[
i_{\text{max}}(G) = \max_{U \text{ independent in } G} \frac{|U|}{|U| + |N_G(U)|}
\]

\[
i^*_\text{max}(G) = \begin{cases} i_{\text{max}}(G), & \text{if } i_{\text{max}}(G) \leq \frac{1}{2} \ \\ 1, & \text{if } i_{\text{max}}(G) > \frac{1}{2} \end{cases}
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Questions of Alon and Lubetzky

\[
i(G) \leq i_{max}(G) \leq i^*(G) \leq A(G)
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**Question** (Alon, Lubetzky - 2007.):
Does every graph \( G \) satisfy \( A(G) = i^*_{max}(G) \)?
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It easily follows from the inequality

\[ i^*_{\max}(G \times H) \leq \max\{i^*_{\max}(G), i^*_{\max}(H)\}. \]
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**Proposition** (weaker inequality): \( i(G \times H) \leq \max\{i^*_{\text{max}}(G), i^*_{\text{max}}(H)\} \)
Consequences

**Conjecture (BNR):** \( A(G \cup H) = \max\{A(G), A(H)\} \), where \( A \cup G \) denotes the disjoint union of \( G \) and \( H \).
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For any rational \( r \in (0, \frac{1}{2}] \cup \{1\} \) there exists a graph \( G \) with \( A(G) = r \).

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From \( A(G) = i^*_{\text{max}}(G) \) we obtain that:
\[
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\]

\( A(G) \) cannot be irrational.
Algorithmic aspects

**Question (BNR):** Is $A(G)$ computable? And if so, what is its complexity?
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If $G$ is bipartite then $A(G)$ can be determined in polynomial time.

From $A(G) = i^\ast \max(G)$ we also obtain that:

The problem of deciding whether $A(G) > t$ for a given graph $G$ and a value $t$, is NP-complete.
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Determining whether $A(G) = 1$ or $A(G) \leq \frac{1}{2}$ can be also done in polynomial time.
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The Hedetniemi conjecture

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For every graph $G$ and $H$ we have

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The fractional version of the conjecture:
($\chi_f$ denotes the fractional chromatic number of the graph.)

$$\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}.$$  

$\chi_f(G \times H) \leq \min\{\chi_f(G), \chi_f(H)\}$ is easy.
Tardif, 2005.: $\chi_f(G \times H) \geq \frac{1}{4} \min\{\chi_f(G), \chi_f(H)\}$. 

Theorem (Zhu - 2010.): The fractional version of Hedetniemi’s conjecture is true.

Corollary: The Burr-Erdős-Lovász conjecture is true.
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For any independent set $U$ of $G \times H$ can be partitioned into the union of $A$ and $B$, where for $\forall y \in V(H)$ the projection of the $y$-slice of $A$ is independent in $G$, for $\forall x \in V(G)$ the projection of the $x$-slice of $B$ is independent in $H$; furthermore if $MA$ denotes the $G$-neighborhood of $A$, and $MB$ denotes the $H$-neighborhood of $B$, $A$, $B$, $MA$, $MB$ are pairwise disjoint subsets of $V(G \times H)$. 
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\[ MB = \{ (x, y) \in V(G \times H) : \exists (x, y') \in B, \{ y, y' \} \in E(H) \} \]

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Zhu’s lemma $\Rightarrow i(G \times H) \leq \max\{i^*_{\max}(G), i^*_{\max}(H)\}$:
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Zhu’s lemma $\Rightarrow i(G \times H) \leq \max\{i_{max}^*(G), i_{max}^*(H)\}$:

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Zhu's lemma $\Rightarrow i(G \times H) \leq \max\{i^*_\text{max}(G), i^*_\text{max}(H)\}$:

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$|A| + |B| = |U|, \quad |A| + |B| + |MA| + |MB| \leq |V(G \times H)|$
Thank you for your attention!