Extending the parking space

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(joint with Andrew Berget)

UCSD
A parking function of size $n$ is a labeled Dyck path of size $n$:
- a vertical run of size $k$ is labeled with a subset of $[n]$ of size $k$,
- every letter in $[n]$ appears once as a label.

**Defn**: $\text{Park}_n = \{\text{parking functions of size } n\}$.

**Fact**: $|\text{Park}_n| = (n + 1)^{n-1}$. 
The Parking Space

\( \mathfrak{S}_n \) acts on Park\(_n\) by *label permutation*.

**Q:** How does Park\(_n\) decompose as an \( \mathfrak{S}_n \)-module?
Vertical Run Partitions

If $D$ is a Dyck path of size $n$, get a *vertical run partition* $\lambda(D) \vdash n$.

\[ \lambda(D) = (3, 2, 2) \vdash 7 \]
Coset Decomposition

Given $\lambda \vdash n$, let $\mathcal{S}_\lambda$ be the *Young subgroup*.

$$M^\lambda = \mathcal{S}_n / \mathcal{S}_\lambda = \text{coset representation}.$$  

**Fact:** $\text{Park}_n \cong_{\mathcal{S}_n} \bigoplus_D M^\lambda(D)$, where $D$ ranges over all size $n$ Dyck paths.

**Example:**

Park$_3 \cong_{\mathcal{S}_3} M^{(3)} \oplus 3M^{(2,1)} \oplus M^{(1,1,1)}$. 

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**Diagram:**

[Diagram showing the coset decomposition for Park$_3$]
**Theorem:** [Berget-R] There exists an $\mathfrak{S}_{n+1}$-module $V_n$ such that

\[
\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}(V_n) \cong_{\mathfrak{S}_n} \text{Park}_n.
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Probably not.
Main Theorem

**Theorem:** [Berget-R] There exists an $S_{n+1}$-module $V_n$ such that

$$\text{Res}^{S_{n+1}}_{S_n}(V_n) \cong S_n \text{ Park}_n.$$ 

**Riddle:** Can you see the action of $S_{n+1}$ on Park$_n$?

Probably not.

**Fact:** Park$_n$ does not in general extend to $S_{n+1}$ as a permutation module. Also, Park$_n$ does not in general extend to $S_{n+2}$ at all.
Extendability of Representations

**Problem:** Let $M$ be an $\mathfrak{S}_n$-module. Give a nice criterion for when $M$ extends to $\mathfrak{S}_{n+1}$ (or $\mathfrak{S}_{n+r}$).
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- The irrep $S^\lambda$ extends to $S_{n+1}$ iff $\lambda \vdash n$ is a rectangle minus an outer corner.

- The regular representation $C[S_n]$ extends to $S_{n+2}$.

- The coset representation $M$ does not extend to $S_8$ for $(3,2,2)$.

- The map $\text{Res}: K_0(S_{n+1}) \to K_0(S_n)$ is surjective over $\mathbb{Q}$. 

\[\text{Whitehouse}\]
Extendability of Representations

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- The irrep $S^\lambda$ extends to $\mathfrak{S}_{n+1}$ iff $\lambda \vdash n$ is a rectangle minus an outer corner.

- The regular representation $\mathbb{C}[\mathfrak{S}_n]$ extends to $\mathfrak{S}_{n+2}$. [Whitehouse]
Extendability of Representations

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- The irrep $S^\lambda$ extends to $\mathfrak{S}_{n+1}$ iff $\lambda \vdash n$ is a rectangle minus an outer corner.
- The regular representation $\mathbb{C}[\mathfrak{S}_n]$ extends to $\mathfrak{S}_{n+2}$. [Whitehouse]
- The coset representation $M^\lambda$ does not extend to $\mathfrak{S}_8$ for $\lambda = (3, 2, 2) \vdash 7$. 

The map $\text{Res}: K_0(\mathfrak{S}_{n+1}) \rightarrow K_0(\mathfrak{S}_n)$ is surjective over $\mathbb{Q}$. 
**Problem:** Let $M$ be an $\mathcal{S}_n$-module. Give a nice criterion for when $M$ extends to $\mathcal{S}_{n+1}$ (or $\mathcal{S}_{n+r}$).

- The irrep $S^\lambda$ extends to $\mathcal{S}_{n+1}$ iff $\lambda \vdash n$ is a rectangle minus an outer corner.
- The regular representation $\mathbb{C}[\mathcal{S}_n]$ extends to $\mathcal{S}_{n+2}$. [Whitehouse]
- The coset representation $M^\lambda$ does not extend to $\mathcal{S}_8$ for $\lambda = (3, 2, 2) \vdash 7$.
- The map
  \[ \text{Res} : K_0(\mathcal{S}_{n+1}) \to K_0(\mathcal{S}_n) \]
  is surjective over $\mathbb{Q}$. 

Graphs

\[ K_{n+1} = \text{complete graph on } [n + 1]. \]

A subgraph \( G \subseteq \binom{[n+1]}{2} \) is \textit{slim} if the complement \( K_{n+1} - G \) is connected.
Polynomials

To any subgraph $G \subseteq \left( \binom{[n+1]}{2} \right)$, we associate the polynomial

$$p(G) = \prod_{(i<j) \in G} (x_i - x_j).$$

$$p(G) = (x_2 - x_3)(x_2 - x_6)(x_3 - x_5)(x_3 - x_6)$$
Spaces

**Defn:** Let $V_n \subset \mathbb{C}[x_1, \ldots, x_{n+1}]$ be the subspace

$$V_n = \text{span}\{p(G) : G \subseteq K_{n+1} \text{ is slim}\}.$$

**Obs:** $V_n$ is a graded $\mathfrak{S}_{n+1}$-module.

**Theorem:** [Berget-R] $\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}(V_n) \cong \text{Park}_n$. (Graded structure?)
Defn: The *area* of a Dyck path $D$ is the number of boxes to the northwest of $D$.

\[
\text{area}(D) = 11
\]
**Theorem:** [Berget-R] The $S_n$-isomorphism type of the degree $k$ piece $V_n(k)$ is

$$\bigoplus_{D} M^{\lambda(D)},$$

where $D$ ranges over all size $n$ Dyck paths with area $k$.

**Example:** The graded $S_3$-character of $V_3$ is

$$q^0 M^{(3)} + q^1 M^{(2,1)} + 2q^2 M^{(2,1)} + q^3 M^{(1,1,1)}.$$
Extended Structure

\[ V_n(k) = \text{degree } k \text{ piece of } V_n \text{ for } k = 0, 1, \ldots, \binom{n}{2}. \]
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**Theorem: [Berget-R]** \( V_n(k) \cong \mathfrak{S}_{n+1} \Sym^k(V) \) for \( 0 \leq k < n \), where \( V \) is the reflection representation of \( \mathfrak{S}_{n+1} \).
Extended Structure

\( V_n(k) = \) degree \( k \) piece of \( V_n \) for \( k = 0, 1, \ldots, \binom{n}{2} \).

**Theorem:** [Berget-R] \( \forall k \leq n \), where \( V \) is the reflection representation of \( \mathcal{S}_{n+1} \).

\[ V_n(k) \cong_{\mathcal{S}_{n+1}} \text{Sym}^k(V) \]

**Theorem:** [Berget-R] Let \( C = \langle (1, 2, \ldots, n+1) \rangle \) and \( \zeta = e^{\frac{2\pi i}{n+1}} \). Then

\[ V_n(\text{top}) = V_n \left( \binom{n}{2} \right) \cong_{\mathcal{S}_{n+1}} \text{Ind}_{C}^{\mathcal{S}_{n+1}}(\zeta) \otimes \text{sign}. \]
Open Problems

**Problem:** Given a nice criterion for deciding whether an $S_n$-module $M$ extends to $S_{n+1}$ (or $S_{n+r}$).

**Problem:** Determine the full graded $S_{n+1}$-structure of $V_n$.

**Problem:** For $n$ and $k$ fixed, what is the maximum $r$ so that $V_n(k)$ extends to $S_{n+r}$?

- $k = 0 \Rightarrow r = \infty$
- $k = 1, n > 2 \Rightarrow r = 1$
- $k = \text{top} \Rightarrow r \geq 2$. 
Thanks for listening!

A. Berget and B. Rhoades. Extending the parking space.
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