Parameters of Two-Prover-One-Round Game and The Hardness of Connectivity Problems

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This Talk

- Investigate the **connection** between
  - 2-Prover-1-Round Game (2P1R) and
  - Hardness of Approximating Connectivity Problems

- Investigate recent **technology** in 2P1R
  - 2P1R with small alphabets, degree reduction, random sampling

- **Improve** Hardness of Approximating:
  - Rooted $k$-Connectivity
  - Vertex Connectivity Survivable Network Design (VC-SNDP)
  - Vertex Connectivity $k$-Route Cut (VC $k$-Route Cut)
Motivated by an attempt to improve hardness of Rooted $k$-Connectivity
Background & Motivation

- Hardness factor of approximating Rooted $k$-Connectivity and several problems depend on parameters of the Label-Cover problem (a.k.a, 2P1R).

- Attempts to improve hardness factor require optimizing the parameters of Label-Cover.

- Techniques for optimizing parameters of 2P1R are known in PCP community but not in APPROX community.

PCP = Probabilistic Checkable Proof / APPROX = Approximation Algorithm
Two Communities view things in different ways
Two Communities

PCP Community
(complexity)

- View 2P1R as
  2-Query PCP
- Know:
  • Techniques for optimizing parameters of 2P1R
- Obsecure:
  • Applications of optimizing parameters

Approx Community
(algorithm)

- View 2P1R as
  Label-Cover
- Know:
  • Reductions from Label-Cover to connectivity problems
- Obsecure:
  • PCP techniques and recent progress
2-Prover-1-Round Game (2P1R)

Proof System: 1 Verifier and 2 Provers.
Provers want to convince that a proof is valid.
2-Prover-1-Round Game (2P1R)

Protocol: (1) Verifier asks each prover one question.
2-Prover-1-Round Game (2P1R)

Protocol:
1. Verifier asks each prover one question.
2. Each prover answers the question.
2-Prover-1-Round Game (2P1R)

**Protocol:** The Verifier accepts the proof.

⇔ Two answers are **valid** and **consistent**.

```
Verifier

Give an answer

Prover 1

No Cooperation

Give an answer

Prover 2

Accept / Reject
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Approx Views 2P1R as Label-Cover
Label-Cover

• We wish to color (label) vertices of a bipartite graph to satisfy admissible color pairs (constraint) on each edge.
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The coloring satisfies a constraint on an edge $e$ since Red-Green is admissible.
Label-Cover

- We wish to color (label) vertices of a bipartite graph to satisfy admissible color pairs (constraint) on each edge.

We may need more than one colors on each vertex.
The Cost of Label-Cover

- The cost is the total number of colors used.

Total cost = 12
Hardness of Label-Cover

- **Hardness Depends on Two Parameters**
  - Maximum **Degree** : $D$
  - **Alphabet** Size (# of available colors) : $L$

(We abuse $L$ to mean both the set and its size.)

$D = 2$

$L = 3$
Label-Cover
and
Connectivity Problems
Hardness of Connectivity Problems

- Hardness results of many connectivity problems were derived from Label-Cover.

- Rooted $k$-Conn
  - Cheriyan, L, Naves, Vetta
  - SODA 2012

- VC-SNDP
  - Kortsarz, Krauthgamer, Lee
  - SICOMP 2003
  - Chkrabor, Chuzhoy, Khanna
  - STOC 2008

- VC $k$-Route Cut
  - Chuzhoy, Makarychev, Vijayaraghavan, Zhou
  - SODA 2012
Root $k$-Connectivity

Input
- A graph $G=(V,E)$ with costs on edges
- A root vertex $r$
- A set of terminals $T \subseteq V$

Goal
- Find a min-cost subgraph $H \subseteq G : H$ has $k$-vertex disjoint paths from $r$ to each terminal $t \in T$. 
Root $k$-Connectivity

**Input**
- A graph $G=(V,E)$ with costs on edges
- A root vertex $r$
- A set of terminals $T \subseteq V$

**Goal**
- Find a min-cost subgraph $H \subseteq G : H$ has $k$-vertex disjoint paths from $r$ to each terminal $t \in T$.

We want to **connect** $r$ to $t \in T$ by $k$ vertex-disjoint paths.
Vertex-Connectivity Survivable Network Design (VC-SNDP)

Input
- A graph $G=(V,E)$ with costs on edges
- A requirement $k(s,t)$ for each pair $s,t \in V$

Goal
- Find a min-cost subgraph $H \subseteq G : H$ has $k(s,t)$ vertex-disjoint paths for each pair $s,t \in V$. 
Vertex-Connectivity Survivable Network Design (VC-SNDP)

Input

- A graph $G=(V,E)$ with costs on edges
- A requirement $k(s,t)$ for each pair $s,t \in V$

Goal

- Find a min-cost subgraph $H \subseteq G : H$ has $k(s,t)$ vertex-disjoint paths for each pair $s,t \in V$.

We want to connect each $s,t$ by $k$ vertex-disjoint paths.
Vertex-Connectivity $k$-Route Cut (VC $k$-Route Cut)

Input
- A graph $G=(V,E)$ with costs on edges
- Source-sink pairs $(s_1,t_1), \ldots, (s_q,t_q)$

Goal
- Find a min-cost subset of edges $F \subseteq E$ : $G - F$ has < $k$ vertex-disjoint paths between $s_i,t_i$ for all $i$
Vertex-Connectivity $k$-Route Cut (VC $k$-Route Cut)

Input
- A graph $G=(V,E)$ with costs on edges
- Source-sink pairs $(s_1,t_1), \ldots, (s_q,t_q)$

Goal
- Find a min-cost subset of edges $F \subseteq E : G - F$ has $< k$ vertex-disjoint paths between $s_i,t_i$ for all $i$

We want to cut down connectivity of $s_i,t_i$ to $k-1$. 
Previous Known Hardness

- Rooted $k$-Conn
  - Cheriyan et al., 2012
- VC-SNDP
  - Chkraborty et al., 2008
- VC $k$-Route Cut
  - Chuzhoy et al., 2012

$\varepsilon$ is a very small fixed constant, which is different for each problem.
Where does a factor $k^\varepsilon$ come from?
Parameters Conversion: Label-Cover To Connectivity

Label Cover

Degree = $D$, Alphabet-Size = $L$

Rooted $k$-Conn

Directed: $k = D$
Undirected: $k = D^3L + D^4$

VC-SNDP

Undirected: $k = DL + D^2$

VC $k$-Route Cut

Undirected: $k = DL$
There is $\gamma > 0$ such that, for any $\ell > 0$, it is NP-Hard to approximate an instance of Label-Cover with

$$degree = 2^{O(\ell)} \text{ and } |L| = 2^{O(\ell)}$$

to within a factor of $2^{\gamma \ell}$.
There is $\gamma > 0$ such that, for any $\ell > 0$, it is NP-Hard to approximate an instance of Label-Cover with

$$\text{degree} = 2^{O(\ell)} \text{ and } |L| = 2^{O(\ell)}$$

to within a factor of $2^{\gamma \ell}$

**Hardness factor** $= D^{\varepsilon_1} = L^{\varepsilon_2} : \varepsilon_1, \varepsilon_2 > 0$ are very small constants.
Recent Technologies?
Recent PCP Techniques (Obsecure to Approx)

- **Right Degree Reduction**
  - Moshkovitz-Raz, J.ACM 2010 / FOCS 2008

- **Alphabet Reduction**
  - Dinur-Harsha, FOCS 2009
Recent PCP Techniques (Obsecure to Approx)

• Right Degree Reduction
  – Moshkovitz-Raz, J.ACM 2010 / FOCS 2008
    Title: Two Query PCP with Sub-Constant Error

• Alphabet Reduction
  – Dinur-Harsha, FOCS 2009
    Title: Composition of Low-Error 2-Query PCPs Using Decodable PCPs
Recent Progress on 2P1R (Obsecure to Approx)

• 21PR with small alphabet-size
  – Khot-Safra, FOCS 2011:
    **Label-Cover:** alphabet-size $L = q^6$, hardness factor $q^{1/2}$
    (but \textit{degree} $>> q$)
  
  – Chan, STOC 2013:
    **Label-Cover:** alphabet-size $L = q^2$, hardness factor $q^{1/2}$
    (but \textit{degree} $>> q$)
Recent Progress on 2P1R (Obsecure to Approx)

- 2P1R with small alphabet-size
  - Khot-Safra, FOCS 2011:
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    (but degree $>> q$)
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    (but degree $>> q$)
Improve Hardness by Optimizing Label-Cover Parameters
Modifying Label-Cover Instance

Chan's Label-Cover

\[ G_0 = (U,W;E) : G_0 \text{ is left-regular} \]
(but not right-regular)

Make the graph regular

\[ G_1 : G_1 \text{ is regular} \]
(but degree is large)

Random Sampling

\[ G_2 : G_2 \text{ has small degree} \]
max-degree \(\approx\) hardness
Modifying Label-Cover Instance

Chan's Label-Cover
\[ G_0 = (U,W;E) : G_0 \text{ is } \text{left-regular} \]
(but not right-regular)

Make the graph regular
\[ G_1 : G_1 \text{ is } \text{regular} \]
(but degree is large)

Random Sampling
\[ G_2 : G_2 \text{ has small degree} \]
max-degree \( \approx \) hardness

Each step must preserves Hardness Factor
Make the graph regular

Chan's Label-Cover
$G_0 = (U,W;E)$: $G_0$ is left-regular
(but not right-regular)

Right Degree Reduction
$G_{1,1}$: $G_{1,1}$ is $(d_1,d_2)$-regular
(left-deg $d_1$, right-deg $d_2$, $d_1 > d_2$)

Make Copies of Left Vertices
$G_{1,2}$: $G_{1,2}$ is $d_1$-regular
Random Sampling

Regular Graph
\( G_1 : G_1 \) is \( d \)-regular
(but degree is large)

Sampling Edges with \( Pr = \frac{D}{d} \)
\( G_{2,1} : G_{2,1} \) has avg-deg \( D \)
(but max-degree is large)

Remove Vertices with Deg > 2D
\( G_{2,2} : G_{2,2} \) has max-deg \( \leq 2D \)
(Set \( D \) = Hardness Factor)
Final Results:

- **Label Cover**
  - Max Degree = $2q$
  - Alphabet-Size = $q^2$
  - Hardness Factor $q^{1/2}$

- **Rooted $k$-Conn**
- **VC-SNDP**
- **VC $k$-Route Cut**

**Hardness Factor**

- Directed: $k^{1/2}$
- Undirected: $k^{1/10}$
- Unirected: $k^{1/8}$
- Unirected: $k^{1/6}$
Hardness in Other Parameter

• Let $P = \#$ of source-sink pairs.

• When $P < k$, The best know approx for Rooted k-Conn, VC-SNDP are $P$-approx by trivial algorithms.

• **Hardness Technique:** Partition edges of the Label-Cover instance into induced matchings by **strong-edge coloring**.

  $\Rightarrow$ Hardness of $P^{1/4}$ for Rooted k-Conn, VC-SNDP and VC k-Route Cut
Conclusion

- We investigate the relation between Label-Cover and connectivity problems.

- We improve hardness of connectivity problems by modifying Label-Cover Instance.

- The hardness in terms of # of source-sink can be obtained by graph-coloring technique.
Thank you for your attention.