The Width of “Canonical” Trees and of Acyclic Digraphs

Daniel Krenn
(joint work with Clemens Heuberger and Stephan Wagner)

Graz University of Technology, Austria

June 11, 2013
Partitions of 1 into Powers of $t$

- integer base $t \geq 2$

**Definition**

$C_{\text{Partition}}$ is the set of integer tuples $(x_1, \ldots, x_r)$ for which

- $0 \leq x_1 \leq x_2 \leq \cdots \leq x_r$,
- $1 = \sum_{i=1}^{r} \frac{1}{t^{x_i}}$. 
Partitions of 1 into Powers of $t$

- integer base $t \geq 2$

**Definition**

$C_{\text{Partition}}$ is the set of integer tuples $(x_1, \ldots, x_r)$ for which

- $0 \leq x_1 \leq x_2 \leq \cdots \leq x_r$,
- $1 = \sum_{i=1}^{r} \frac{1}{t^{x_i}}$.

E.g. $t = 2$, partitions with 5 summands:

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} = \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8}.$$
Canonical Compact $t$-ary Prefix Codes

- Integer $t \geq 2$

Lemma (Kraft–McMillan inequality)

$$\sum_{c \in C} \frac{1}{t^{\text{length}(c)}} \leq 1.$$
Canonical Compact $t$-ary Prefix Codes

- integer $t \geq 2$

**Lemma (Kraft–McMillan inequality)**

$$\sum_{c \in C} \frac{1}{t^{\text{length}(c)}} \leq 1.$$  

**Definition**

$C_{\text{Code}}$ is the set of prefix codes $C \subseteq \{0, \ldots, t - 1\}^*$ for which

- $C$ **compact** (equality in Kraft–McMillan inequality)
- $C$ **canonical** (lexicographic ordering of its words corresponds to a non-decreasing ordering of the word lengths)
Canonical Compact $t$-ary Prefix Codes

- integer $t \geq 2$

Lemma (Kraft–McMillan inequality)

$$\sum_{c \in C} \frac{1}{t^{\text{length}(c)}} \leq 1.$$ 

Definition

$C_{Code}$ is the set of prefix codes $C \subseteq \{0, \ldots, t-1\}^*$ for which

- $C$ compact (equality in Kraft–McMillan inequality)
- $C$ canonical (lexicographic ordering of its words corresponds to a non-decreasing ordering of the word lengths)

E.g. $t = 2$, codes of size 5:

$\{0, 10, 110, 1110, 1111\}$, $\{0, 100, 101, 110, 111\}$, $\{00, 01, 10, 110, 111\}$. 
Canonical Trees

- integer $t \geq 2$

**Definition (Rooted $t$-ary Tree)**
- one vertex has been designated as the root
- all vertices have $t$ ("inner vertex") or no ("leaf") successors

**Definition (Canonical Rooted $t$-ary Tree)**
- the longer paths are as far to the right hand side as possible
  (also called level-greedy tree)

Only the last tree is canonical.
Partitions, Codes and Trees

Identification

\[ C_{\text{Partition}} = C_{\text{Code}} = C_{\text{Tree}} \]
Number of Canonical Trees

- asymptotic formula (main term)
  \[
  \sim R\rho^n
  \]
  with \( \rho \to 2 \) for increasing \( t \)
Number of Canonical Trees

- asymptotic formula (main term)
  \[ \sim R\rho^n \]
  with \( \rho \to 2 \) for increasing \( t \)
- more precisely
  \[ R\rho^n + R_2\rho_2^n + O(r_3^n) \]
  with
  - \( \rho > \rho_2 > r_3, R, R_2 \) positive real constants depending on \( t \)
  - asymptotic expansions (as \( t \to \infty \))
    - \( \rho = 2 - \frac{1}{2^{t+1}} + O\left(\frac{t}{2^{2t}}\right) \)
    - \( R = \frac{1}{8} + \frac{t - 2}{2^{t+5}} + O\left(\frac{t^2}{2^{2t}}\right) \)
    - \( \rho_2 = \ldots, R_2 = \ldots, r_3 = \ldots, \)
  - explicit \( O \)-constants
  (Elsholtz–Heuberger–Prodinger 2012)
### Theorem (Heuberger–K–Wagner 2012)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>height</strong></td>
<td>asymptotically normally distributed</td>
</tr>
<tr>
<td></td>
<td>mean $\sim \mu_h n$</td>
</tr>
<tr>
<td></td>
<td>variance $\sim \sigma_h^2 n$</td>
</tr>
<tr>
<td><strong>number of distinct summands</strong></td>
<td>asymptotically normally distributed</td>
</tr>
<tr>
<td></td>
<td>mean $\sim \mu_d n$</td>
</tr>
<tr>
<td></td>
<td>variance $\sim \sigma_d^2 n$</td>
</tr>
<tr>
<td><strong>total path length</strong></td>
<td>asymptotically normally distributed</td>
</tr>
<tr>
<td></td>
<td>mean $\sim \mu_{tpl} n^2$</td>
</tr>
<tr>
<td></td>
<td>variance $\sim \sigma_{tpl}^2 n^3$</td>
</tr>
<tr>
<td><strong>maximum number of equal summands</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>mean $\mu_w \log n + O(\log \log n)$</td>
</tr>
<tr>
<td></td>
<td>concentration property</td>
</tr>
<tr>
<td><strong>number of leaves on the last level</strong></td>
<td>discrete limit distribution</td>
</tr>
<tr>
<td></td>
<td>mean $2t + o(1)$</td>
</tr>
<tr>
<td></td>
<td>variance $2t^2 + o(1)$</td>
</tr>
<tr>
<td><strong>number of leaves</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>discrete limit distribution</td>
</tr>
<tr>
<td></td>
<td>mean $2t + o(1)$</td>
</tr>
<tr>
<td></td>
<td>variance $2t^2 + o(1)$</td>
</tr>
</tbody>
</table>

Analysis of Parameters of Trees Corresponding to Huffman Codes

Daniel Krenn, TU Graz, Austria
Theorem (Heuberger–K–Wagner 2012)

- random canonical tree $T$ of size $n$
- width has expectation

$$\mathbb{E}(w(T)) = \mu_w \log n + O(\log \log n)$$

with

$$\mu_w = \frac{1}{-(t - 1) \log q_0} = \frac{1}{t \log(2)} + \frac{1}{t^2 \log(2)} + O\left(\frac{1}{t^2}\right)$$

- concentration property

$$\mathbb{P}(|w(T) - \mu_w \log n| \geq 3\mu_w \log \log n) = O\left(\frac{1}{\log n}\right)$$
Labelled Acyclic Digraphs

Figure: An acyclic digraph with height 3 and width 4.

- representation with “layers”
- height $\leftrightarrow$ number of layers (length of a longest path)
- top layer: sources
- bottom layer: sinks
- width is maximal number of vertices at one layer
Some Properties

- **number** of labelled acyclic digraphs
- asymptotic formula

\[ \sim C \rho^n 2^{\binom{n}{2}} n! \]

with positive constants $C, \rho$

Some Properties

- **number** of labelled acyclic digraphs
  - asymptotic formula
    \[ \sim C\rho^n 2^{\binom{n}{2}} n! \]
  - with positive constants $C, \rho$

- **height** of an acyclic digraph
  - asymptotically normally distributed
  - mean $\sim \mu_h n$
  - variance $\sim \sigma_h^2 n$
    (McKay 1989)
The Width of Acyclic Digraphs

Theorem (K–Wagner 2013)

- **random (labelled) acyclic digraph** $D$ of size $n$
- **width has expectation**

\[
\mathbb{E}(w(D)) = \sqrt{\log S} n + O(\log \log n)
\]

with $S = \sqrt{2}$

- **concentration property**

\[
\mathbb{P}\left( \left| w(D) - \sqrt{\log S} n \right| \geq \log_S \log_S n \right) = O\left( \frac{1}{\sqrt{\log S} n^{3\sqrt{\log S} n}} \right)
\]
Infinite Transfermatrix

- building an acyclic digraph
  - with $r$ sinks
  - out of a one with $s$ sinks
  - by adding a layer

\[ M_\infty = \begin{pmatrix} \cdots & \cdots \frac{q^r}{2(\frac{r}{2})_r} (1 - 2^{-s})^r & \cdots \\ \cdots & \cdots \frac{q^r}{2(\frac{r}{2})_r} (1 - 2^{-s})^r & \cdots \\ \cdots & \cdots \frac{q^r}{2(\frac{r}{2})_r} (1 - 2^{-s})^r & \cdots \end{pmatrix} \]
Infinite Transfermatrix

- building an acyclic digraph
  - with $r$ sinks
  - out of a one with $s$ sinks
  - by adding a layer
- leads to recursion
- generating function

$$G(q) = \sum_{\text{$D$ acyclic digraph}} \frac{q^n}{2\binom{n}{2} n!}$$

equals

$$\sum_{\text{all matrix entries}} (I - M_\infty)^{-1} S$$

- $s = 3$
- $r = 2$

$$M_\infty = \begin{pmatrix} \cdots & \cdots & q^r \frac{1}{2\binom{r}{2} r!} \left(1 - 2^{-s}\right)^r \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

Analysis of Parameters of Trees Corresponding to Huffman Codes

Daniel Krenn, TU Graz, Austria
Width-Restrictions

- restrict to acyclic digraphs with width \( \leq W \)
- \( \leadsto \) finite transfer matrix

\[
M_W = \begin{pmatrix}
\cdots & \cdots & \frac{q^r}{2^\left(\binom{r}{2}\right) r!} (1 - 2^{-s})^r & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

- \( \leadsto \) generating function

\[
G_W(q) = \sum_{D \text{ acyclic digraph with width } \leq W} \frac{q^n}{2^{\binom{n}{2}} n!} = \sum \text{ all matrix entries } \left( I - M_W \right)^{-1} S
\]
Main Steps in the Proof

- generating function $G(q)$, $G_W(q)$
- dominant singularities $q_\infty$, $q_W$
- singularity analysis

$$\mathbb{P}(\text{width of acyclic digraph} \leq W) \sim c_W \left( \frac{q_W}{q_\infty} \right)^{-n-1}$$

with
- $c_W \to 1$ (comes from finite/infinite determinants)
Main Steps in the Proof

- generating function $G(q)$, $G_W(q)$
- dominant singularities $q_\infty$, $q_W$
- singularity analysis

\[
\mathbb{P}(\text{width of acyclic digraph} \leq W) \sim c_W \left( \frac{q_W}{q_\infty} \right)^{-n-1}
\]

with

- $c_W \to 1$ (comes from finite/infinite determinants)
- expectation

\[
\mathbb{E}(\text{width of acyclic digraph}) = \sum_{W \geq 0} (1 - \mathbb{P}(\text{width} \leq W))
\]

- splitting up the sum into
Dominant Singularities

- generating function

\[ G(q) = \sum_{\text{all matrix entries}} (I - M_\infty)^{-1} S \]

with (infinite) transfer matrix \( M_\infty \)

- dominant singularity \( \leftrightarrow \) eigenvalue 1 of \( M_\infty \)
Dominant Singularities

- generating function

\[ G(q) = \sum_{\text{all matrix entries}} (I - M_\infty)^{-1} S \]

with (infinite) transfer matrix \( M_\infty \)

- dominant singularity \( \leftrightarrow \) eigenvalue 1 of \( M_\infty \)
- infinite positive eigenvector
- truncate and apply methods from Perron–Frobenius theory
Dominant Singularities

- generating function

$$G(q) = \sum_{\text{all matrix entries}} \frac{1}{(I - M_\infty)^{-1}} S$$

with (infinite) transfer matrix $M_\infty$

- dominant singularity $\leftrightarrow$ eigenvalue 1 of $M_\infty$
- infinite positive eigenvector
- truncate and apply methods from Perron–Frobenius theory
- $q_W$ converges to $q_\infty$ with rate $\sqrt{2 - W^2}$
Theorem (K–Wagner 2013)

- random (labelled) acyclic digraph $D$ of size $n$
- width has expectation

$$\mathbb{E}(w(D)) = \sqrt{\log S} n + O(\log \log n)$$

with $S = \sqrt{2}$

- concentration property

$$\Pr \left( \left| w(D) - \sqrt{\log S} n \right| \geq \log S \log S n \right) = O \left( \frac{1}{\sqrt{\log S n^{3/2} \log S n}} \right)$$