Bi-angular lines in $\mathbb{R}^n$

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Joint work with Darcy Best

University of Lethbridge

_CanaDAM 2013_

Memorial University of Newfoundland

June 10 – 13, 2013
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Mutually unbiased weighing matrices
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- Mutually unbiased weighing matrices
- Mutually Suitable Latin Squares
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- The auxiliary matrices corresponding to weighing matrices
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Biangular lines from orthogonal blocks
Biangular lines and association schemes
Bi-angular lines in $\mathbb{R}^n$

Let $V$ be a set of unit vectors in $\mathbb{R}^n$. $V$ is said to consist of bi-angular lines if $|\langle u, v \rangle| \in \{0, \alpha\}$ for all $u$ and $v$ in $V$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product in $\mathbb{R}^n$ and $0 < \alpha < 1$.

Two Hadamard matrices $H$ and $K$ of order $n$ are called unbiased if all the entries of $HK^*$ have modulus $\sqrt{n}$.

A set $M$ of Hadamard matrices of order $n$ is called mutually unbiased (MU) if every pair of Hadamard matrices in $M$ are unbiased.

Any set of MU Hadamard matrices of order $n$ forms a set of bi-angular lines in $\mathbb{R}^n$ with $\alpha = \frac{1}{\sqrt{n}}$. 
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Mutually unbiased weighing matrices

Definition
A matrix $W = [w_{ij}]$ of order $n$ and $w_{ij} \in \{-1, 0, 1\}$ is called a \textit{weighing matrix with weight $p$} if $WW^t = pI_n$, where $I_n$ is the identity matrix of order $n$. 

A $W(n, n)$ is a Hadamard matrix of order $n$. 

Two weighing matrices $W_1, W_2$ of order $n$ and weight $p$ are called \textit{unbiased} if $W_1W_2^t = \sqrt{p}W$, where $W$ is a weighing matrix of order $n$ and weight $p$. 

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Any set of MU weighing matrices of order \( n \) and weight \( p \) forms a set of bi-angular lines in \( \mathbb{R}^n \) with \( \alpha = \frac{1}{\sqrt{p}} \).
Theorem: Let $m$ be the number of bi-angular lines in $\mathbb{R}^n$. 

The DGS upper bound

Theorem: Let $m'$ be the number of MU Hadamard matrices of order $n$. 

The two upper bounds differ by one for $n = 4k^2$, the order of a Hadamard matrix ($\alpha = \frac{1}{2}k$).
The DGS upper bound

**Theorem:** Let $m$ be the number of bi-angular lines in $\mathbb{R}^n$. Then

$$m \leq \begin{cases} \frac{n(n+2)(1-\alpha^2)}{3-(n+2)\alpha^2} & \text{if } 3-(n+2)\alpha^2 > 0, \\ \frac{n(n+1)(n+2)}{6} & \text{otherwise.} \end{cases}$$
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**Theorem:** Let $m'$ be the number of MU Hadamard matrices of order $n$. Then

$$m' \leq \frac{n}{2}.$$  

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Bi-angular lines with $\alpha = \frac{1}{2}$ in $\mathbb{R}^n$
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Finding bi-angular lines is a challenging problem in general.
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Let $m$ be the DGS upper bound for $\alpha = \frac{1}{2}$. 

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Bi-angular lines with $\alpha = \frac{1}{2}$ in $\mathbb{R}^n$

Finding bi-angular lines is a challenging problem in general.

Let $m$ be the DGS upper bound for $\alpha = \frac{1}{2}$. Then

$$m = \frac{3n(n+2)}{10-n}$$
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Mutually unbiased $W(n, p)$ with $\alpha = \frac{1}{2}$ in $\mathbb{R}^n$

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Motivation

The identity matrix is unbiased with the 14 \(\text{MUW}_8\) W's.

The perpendicularity graph of the Gram matrix of the 120 vectors is the adjacency matrix of an SRG(120, 63, 36, 30). The vertices are a disjoint union of 15 cliques of size 8, forming the point line graph of a \(\text{pg}(7, 8, 4)\) having an automorphism group of size 348,364,800; and may be the same \(\text{pg}\) found by Cohen in 1981.

The identity matrix is unbiased with the 8 \(\text{MUW}_7\) W's.

The perpendicularity graph of the Gram matrix of the 63 vectors is an SRG(63, 30, 13, 15). The vertices are disjoint union of 9 cliques of size 7. The graph is isomorphic to the classical design having as blocks the hyperplanes in \(\text{PG}(5, 2)\).
Motivation

The identity matrix is unbiased with the 14 MUW $W(8, 4)$’s.

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Mutually suitable Latin squares

Two Latin squares $L_1$ and $L_2$ of size $n$ on symbol set \{0, 1, 2, ..., $n-1$\} are called suitable if every superimposition of each row of $L_1$ on each row of $L_2$ results in only one element of the form $(a, a)$.

MSLS (Mutually Suitable Latin Squares) of size $n$ is a special form of MOLS (Mutually Orthogonal Latin Squares) of size $n$.

There are $p-1$ MSLS of size $p$ for each prime power $p$. 
Mutually suitable Latin squares

Two Latin squares $L_1$ and $L_2$ of size $n$ on symbol set \{0, 1, 2, \ldots, n - 1\} are called **suitable** if every superimposition of each row of $L_1$ on each row of $L_2$ results in only one element of the form $(a, a)$. 

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The auxiliary matrices corresponding to weighing matrices

Theorem

There is a weighing matrix $W(n, p)$ if and only if there are $n$ auxiliary $(0, \pm 1)$-matrices $C_0, C_1, C_2, \ldots, C_{n-1}$ of order $n$ such that:

$C^*i = C_i$

$C_i C^*_j = 0, i \neq j$

$C^*_2 i = pC_i$

$C_0 + C_1 + C_2 + \cdots + C_{n-1} = p^2 I_n$

Proof.

Let $r_i$ be the $(i+1)$-th row of $W$ and $C_i = r_t r_i, i = 0, 1, \ldots, n-1$. 
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There is a weighing matrix $W(n, p)$ of order $n$ and weight $p$ if and only if there are $n$ auxiliary $(0, \pm 1)$-matrices $C_0, C_1, C_2, \ldots, C_{n-1}$ of order $n$ such that:

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MU weighing matrices from orthogonal blocks

Starting with a $W(n,p)$, we do the following:

▶ Construct the $n$ auxiliary matrices $C_0, C_1, C_2, \ldots, C_{n-1}$.

▶ Let $L_1, L_2, \ldots, L_q$ be a set of MSLS on the set $\{0, 1, 2, \ldots, m-1\}$, $m \geq n$.

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Lemma: If there is a $W(n,p)$ and $q$ MSLS of size $m$, $m \geq n$. Then there are $q$ mutually unbiased weighing matrices (MUWM), $W(nm,p^2)$. 
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An example of MU weighing matrices

Let \( W = \begin{bmatrix}
0 & 1 & 1 & 1 \\
-1 & 0 & 1 & -1 \\
-1 & -1 & 0 & 1 \\
-1 & -1 & -1 & 0
\end{bmatrix} \).

\( C_0 = 
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
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0 & 1 & 1 & 1
\end{bmatrix}, \)

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\[ W_1 = \begin{pmatrix} C_0 & C_3 & C_1 & 0 & C_2 \\ C_2 & C_0 & C_3 & C_1 & 0 \\ 0 & C_2 & C_0 & C_3 & C_1 \\ C_1 & 0 & C_2 & C_0 & C_3 \\ C_3 & C_1 & 0 & C_2 & C_0 \end{pmatrix} \quad W_2 = \begin{pmatrix} C_0 & C_1 & C_2 & C_3 & 0 \\ 0 & C_0 & C_1 & C_2 & C_3 \\ C_3 & 0 & C_0 & C_1 & C_2 \\ C_2 & C_3 & 0 & C_0 & C_1 \\ C_1 & C_2 & C_3 & 0 & C_0 \end{pmatrix}, \]

\[ W_3 = \begin{pmatrix} C_0 & C_2 & 0 & C_1 & C_3 \\ C_3 & C_0 & C_2 & 0 & C_1 \\ C_1 & C_3 & C_0 & C_2 & 0 \\ 0 & C_1 & C_3 & C_0 & C_2 \\ C_2 & 0 & C_1 & C_3 & C_0 \end{pmatrix}, \]

\[ W_4 = \begin{pmatrix} C_0 & 0 & C_3 & C_2 & C_1 \\ C_1 & C_0 & 0 & C_3 & C_2 \\ C_2 & C_1 & C_0 & 0 & C_3 \\ C_3 & C_2 & C_1 & C_0 & 0 \\ 0 & C_3 & C_2 & C_1 & C_0 \end{pmatrix}. \]

\( W_1, W_2, W_3, W_4 \) form a set of four MUWM of order 20 and weight 9.
Biangular lines from orthogonal segments

Starting with a $W(n, n)$ we do the following:

- Construct the $n$ auxiliary matrices $C_0, C_1, C_2, \ldots, C_{n-1}$.

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Lemma: If there is a $W(n, n)$ and $q$ MSLS of size $m$ on the set $\{0, 1, 2, \ldots, m-1\}$, $m \geq n-1$. Then there are $mnq$ biangular lines in $\mathbb{R}^{mn}$. 
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Biangular lines and association schemes

We have a number of examples where the Gram matrix of biangular lines form 3, 4, 5 and 6-association schemes.

For example:

▶ From a Hadamard matrix of order 4 and the first construction, we have a 5-association schemes on 64 points that collapses to an SRG.

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From this construction, we were able to use small orders of Hadamard matrices and MSLS to generate large 6-association schemes.

\[ \mathcal{H}_{12} \oplus \text{MSLS}(11) \rightarrow \text{AS}(1452; 600, 600, 120, 120, 6, 5) \]

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More applications of biangular lines
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The existence of a specific class of MUWM is equivalent to the existence of MOLS:

There are \( m \) MU Hadamard matrices of order \( 4n^2 \) constructible from \( 2n \) symmetric orthogonal blocks of size \( 2n \) if and only if there are \( m \) MOLS of size \( 2n \).

There is a natural connection between biangular lines and certain classes of codes. Biangular lines lead to codes with constant weights and designated distances. For example, MU Hadamard matrices of order \( 4n^2 \) can be used to generate Kerdock codes. However, in practice the reverse is done!
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Some open questions

▶ Find an upper bound for the number of flat biangular lines in $\mathbb{R}^n$.

▶ Show that there are 36 biangular lines in $\mathbb{R}^6$, with $\alpha = \frac{1}{2}$.

▶ Find a direct construction for the $2^{2n} \mu$ Hadamard matrices of order $2^{2n} - 1$.

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Thank you Ian!