PERFECT MATCHINGS IN UNIFORM HYPERGRAPHS

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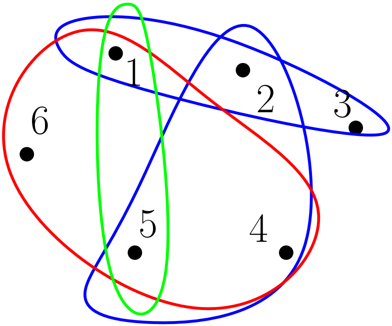
Hypergraphs

A hypergraph H is a family of subsets ($E(H)$) of a ground set $V(H)$

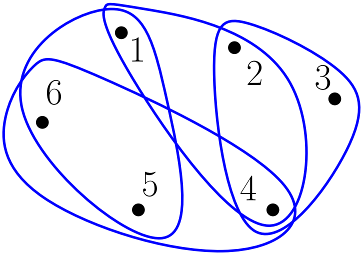
- $H = (V, E)$
- $|V(H)| = n$
- $H := E$



$$V = \{1, 2, 3, 4, 5, 6\}$$
$$E = \{\{1, 5\}, \{1, 2, 3\}, \{2, 4, 5\}, \{1, 4, 5, 6\}\}$$

A hypergraph is k-uniform if all edges are k-sets

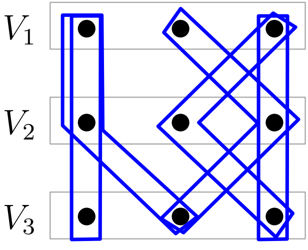
- $H = (V, E)$, $E \subseteq \binom{V}{k}$
- k-graphs
- 2-graphs are graphs



- A k-graph is complete if all k-sets are edges
- $H = (V, E)$, $E = \binom{V}{k}$

$H(V_1, V_2, \ldots, V_k)$ is a k-partite k-graph, if

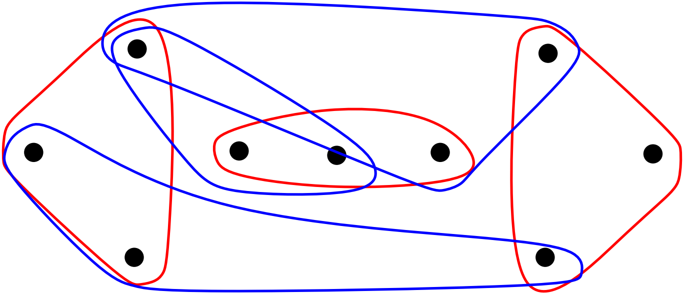
- $V_1, V_2, \ldots, V_k$ is a partition of $V(H)$
- Each edge uses one vertex from each part



- Complete k-partite k-graph
- Balanced complete k-partite k-graph, $K_r(t)$, t : size of each part.

Hypergraphs: Matching

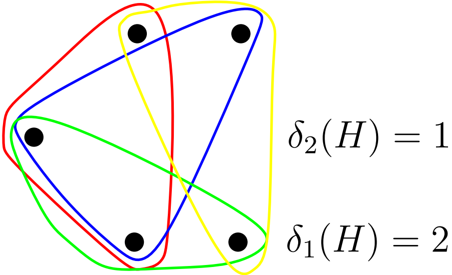
- A matching in a hypergraph is a set of disjoint edges
- A perfect matching is a matching that covers all the vertices
- $\left\lfloor \frac{n}{k} \right\rfloor$ edges in k-graphs
- $n \in k\mathbb{Z}$



Hypergraphs: Degrees

- H: k-graph, $1 \leq d \leq k-1$
- $S \in \binom{V}{d}$
- Degree of S is the number of edges containing S
- $d_H(S) = |\{e \in E : S \subset e\}|$
- minimum d-degree, $\delta_d(H) = \min_{S \in \binom{V}{d}} d_H(S)$

- $d = k-1$: $\delta_{k-1}(H)$
  minimum co-degree
- $d = 1$: $\delta_1(H)$
  minimum vertex degree



Degree Threshold for Perfect Matching

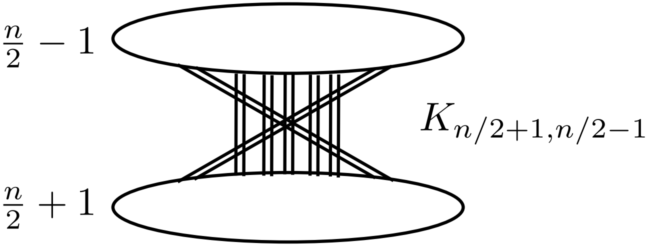
- Sufficient conditions to ensure existence of perfect matching

Definition

$m_d(k, n) = \min\{m : \delta_d(H) \geq m \implies H \text{ has a PM}\}$

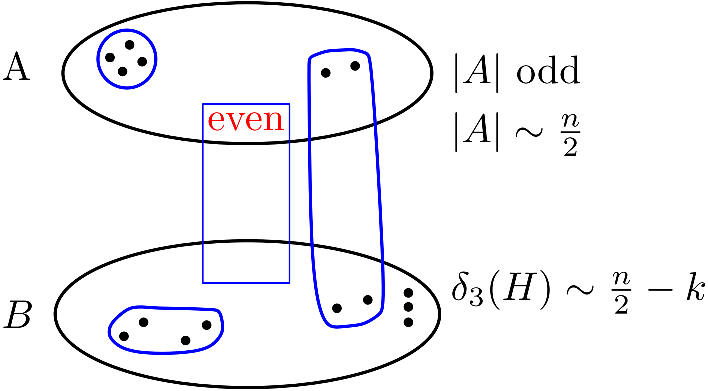
Theorem

$m_1(2, n) \leq \frac{n}{2}$



- Result is best possible: $m_1(2, n) = \frac{n}{2}$.

Perfect Matching: codegree



Perfect Matching: codegree

Theorem

1. *Kühn-Osthus 2006*

   $$m_{k-1}(k,n) \le \frac{n}{2} + 3k^2\sqrt{n \log n}$$

2. *Rödl-Ruciński-Szemerédi 2006*

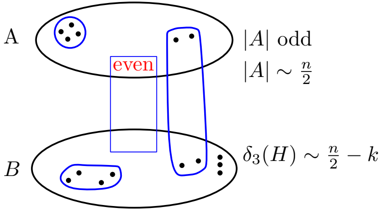
   $$m_{k-1}(k,n) \le \frac{n}{2} + C \log n$$

3. *Rödl-Ruciński-Schacht-Szemerédi 2008*

   $$m_{k-1}(k,n) \le \frac{n}{2} + k/4$$

4. *Rödl-Ruciński-Szemerédi 2009*

   $$m_{k-1}(k,n) \ge \frac{n}{2} - k + \{\tfrac{3}{2}, 2, \tfrac{5}{2}, 3\}$$



Theorem (Pikhurko 2008)

For $\frac{k}{2} \leq d \leq k-1$

$$m_d(k,n) \leq \left(\frac{1}{2}+\epsilon\right)\binom{n-d}{k-d}$$

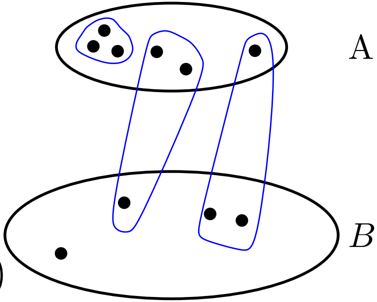
Theorem (Treglown-Zhao 2012)

For $\frac{k}{2} \leq d \leq k-1$

$$m_d(k,n) \sim \frac{1}{2}\binom{n-d}{k-d}$$

Perfect Matching: vertex-degree

$|A| = \frac{n}{3} - 1$



$\delta_1(H) = \binom{n-1}{2} - \binom{2n/3}{2}$

Conjecture

$$1 \le d < k/2 \quad m_d(k,n) \sim \binom{n-d}{k-d} - \binom{n - \frac{n}{k} + 1 - d}{k-d}$$

$$1 \le d < k/2 \quad m_d(k,n) \sim \left(1 - \left(\frac{k-1}{k}\right)^{k-d}\right)\binom{n-d}{k-d}$$

Conjecture

$$1 \leq d < k/2 \quad m_d(k,n) \sim \left(1 - \left(\frac{k-1}{k}\right)^{k-d}\right) \binom{n-d}{k-d}$$

Theorem (Hàn-Person-Schacht 2009)

$$d < \frac{k}{2} \quad m_d(k,n) \leq \left(\frac{k-d}{k} + \epsilon\right) \binom{n-d}{k-d}$$

Theorem (Markström-Ruciński 2010)

$$d < \frac{k}{2} \quad m_d(k,n) \leq \left(\frac{k-d}{k} - \frac{1}{k^{k-1}} + \epsilon\right) \binom{n-d}{k-d}$$

Perfect Matching: Vertex Degree

Conjecture

$$1 \leq d < k/2 \quad m_d(k,n) \sim \left(1 - \left(\frac{k-1}{k}\right)^{k-d}\right)\binom{n-d}{k-d}$$

$k = 3, d = 1 \rightarrow \frac{5}{9}$. $k = 4, d = 1 \rightarrow \frac{37}{64}$. $k = 5, d = 1 \rightarrow \frac{369}{625}$.

Theorem (Hàn-Person-Schacht 2009)

$$m_1(3,n) \leq \left(\frac{5}{9} + \epsilon\right)\binom{n}{2}$$

Theorem (Markström- Ruciński 2010)

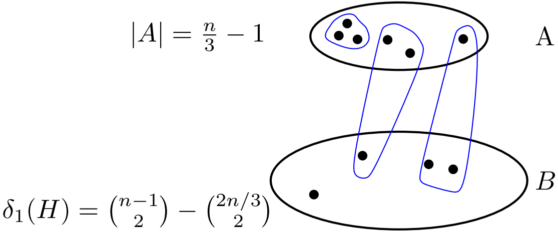
$$m_1(4,n) \leq \left(\frac{42}{64} + \epsilon\right)\binom{n}{3}$$

Theorem (K.)

If H is a 3-graph on $n \geq n_0$ vertices and

$$\delta_1(H) \geq \binom{n-1}{2} - \binom{2n/3}{2} + 1,$$

then H contains a perfect matching.



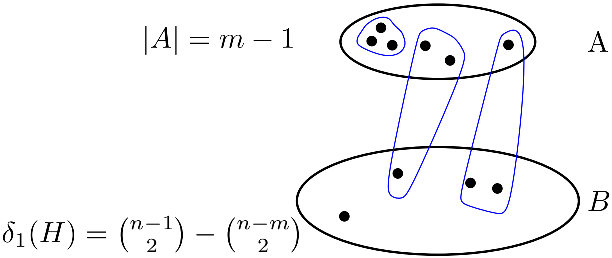
- Independently, Kühn-Osthus-Treglown proved this.
- In fact they proved a stronger result.

Theorem (Kühn-Osthus-Treglown)

If H is a 3-graph on $n \geq n_0$ vertices, $1 \leq m \leq n/3$, and

$$\delta_1(H) \geq \binom{n-1}{2} - \binom{n-m}{2} + 1,$$

then H contains a matching of size at least m.



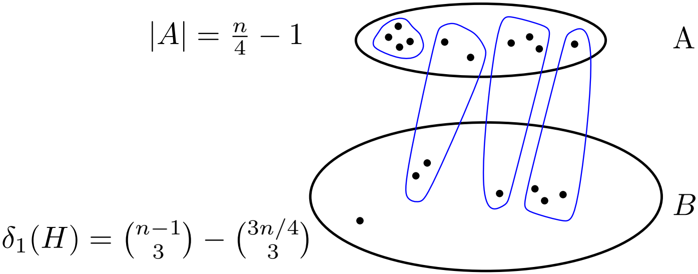
Perfect Matching: Vertex Degree

Theorem (K.)

If H is a 4-graph on $n \geq n_0$ vertices and

$$\delta_1(H) \geq \binom{n-1}{3} - \binom{3n/4}{3} + 1,$$

then H contains a perfect matching.



Perfect Matching: Vertex Degree

Theorem (Alon-Frankl-Huang-Rödl-Ruciński-Sudakov 2012)

- $m_1(4, n) \sim \frac{37}{64}\binom{n-1}{3}$
- $m_2(5, n) \sim \frac{1}{2}\binom{n-2}{3}$
- $m_1(5, n) \sim \frac{369}{625}\binom{n-1}{4}$
- $m_2(6, n) \sim \frac{671}{1296}\binom{n-2}{4}$
- $m_3(7, n) \sim \frac{1}{n}\binom{n-3}{3}$

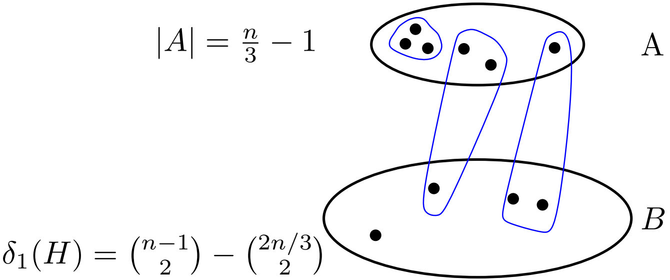
Perfect Matching: Vertex Degree

Theorem (K.)

If H is a 3-graph on $n \geq n_0$ vertices and

$$\delta_1(H) \geq \binom{n-1}{2} - \binom{2n/3}{2} + 1,$$

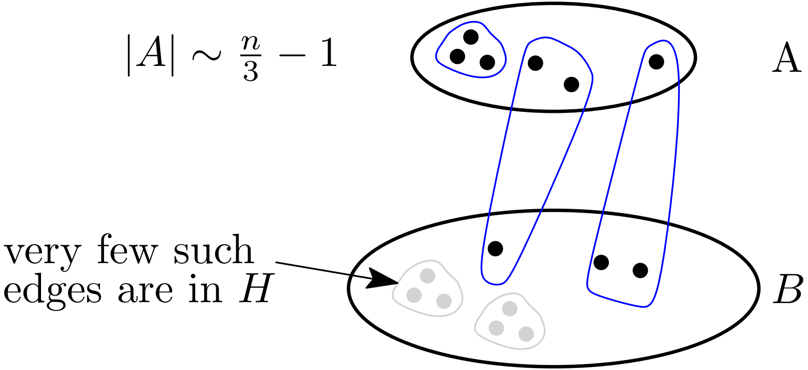
then H contains a perfect matching.



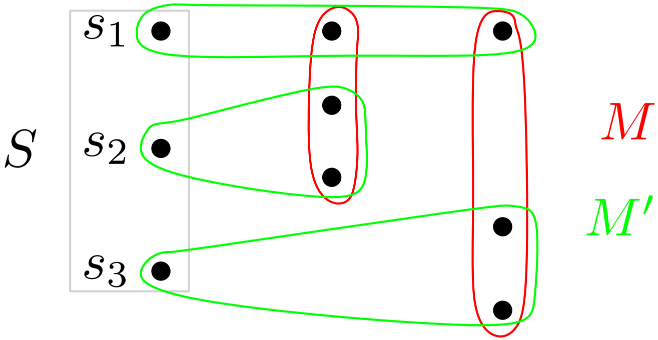
3-graphs - vertex degree: Proof Strategy

We consider two cases

1. H is close to the extremal construction
2. H is non-extremal



- Absorbing Technique
- $S \subset V$, A matching M absorbs the set S if
  $\exists \; M' : V(M') = V(M) \cup S$.



Absorbing Lemma (Hàn-Person-Schacht 2009)

If $\delta_1(H) \geq \left(\frac{1}{2} + \epsilon\right) \binom{n}{k}$, then

$\exists \; M_A$ such that

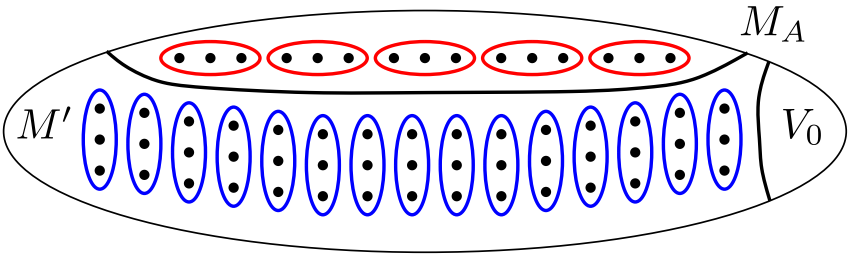
- $|V(M_A)| = \epsilon_1 n$ and
- $\forall \; S \; : \; |S| = \epsilon_2 n$, M_A is S-absorbing.

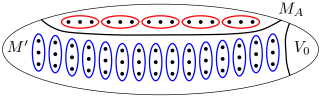
Proof Outline:

1. Find a small absorbing matching M_A ($|V(M_A)| \leq \epsilon_1 n$)
2. Find an almost perfect matching M' in $H - V(M_A)$

   $V_0 = V(H) - (V(M_A) + V(M'))$

   $|V_0| \leq \epsilon_2 n$
3. Absorb V_0 into M_A



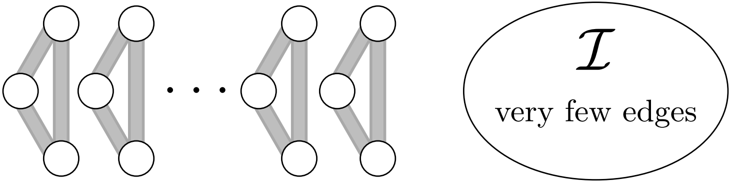


- We cover almost all graph with complete tripartite graphs

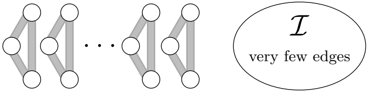
Theorem (Erdős 1964)

If $|E(H)| \geq \epsilon\binom{n}{3}$, then H has $K_3(c\sqrt{\log n})$.

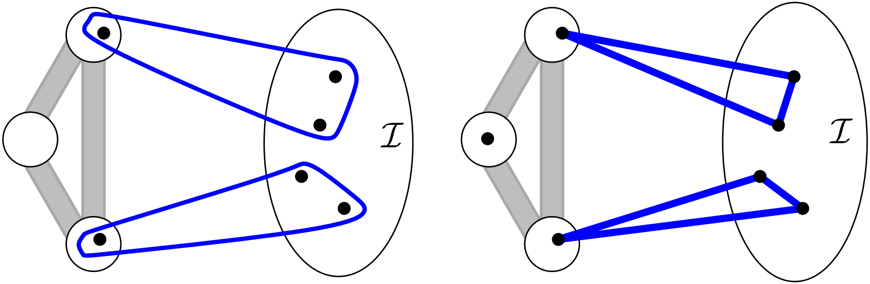
- Using this find as many $K_3(t)$'s.



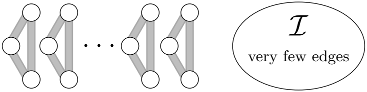
- Extend this to almost perfect cover.



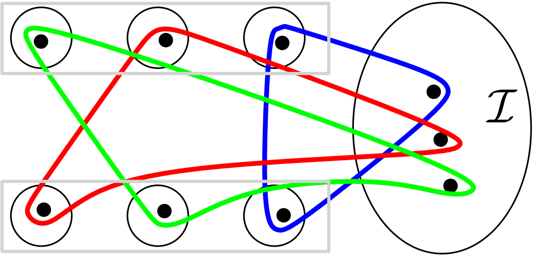
- Extend this to almost perfect cover.
- Suppose many pairs in $\mathcal{I}$ make edges with many vertices in two color classes of many tripartite graphs.

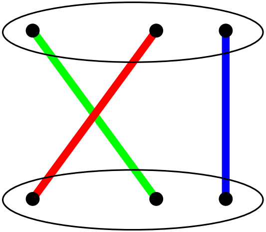


3-graphs - vertex degree: Almost perfect cover



- Extend this to almost perfect cover.
- The link graph:

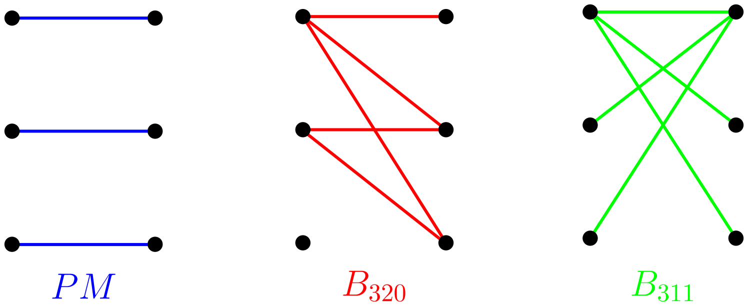




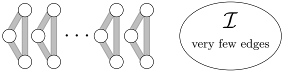
Fact

Let B be a balanced bipartite graph on 6 vertices. If B has at least 5 edges, then

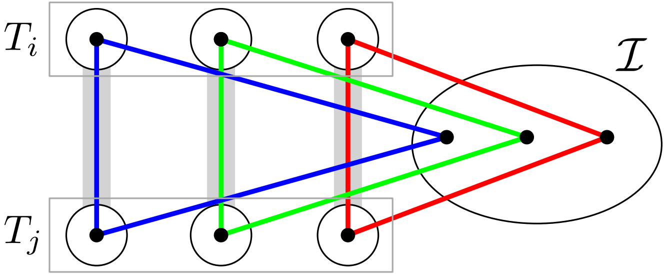
- *B has a perfect matching* or
- *B contains B_{320} as a subgraph* or
- *B is isomorphic to B_{311}.*

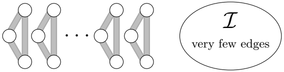


3-graphs - vertex degree: Almost perfect cover

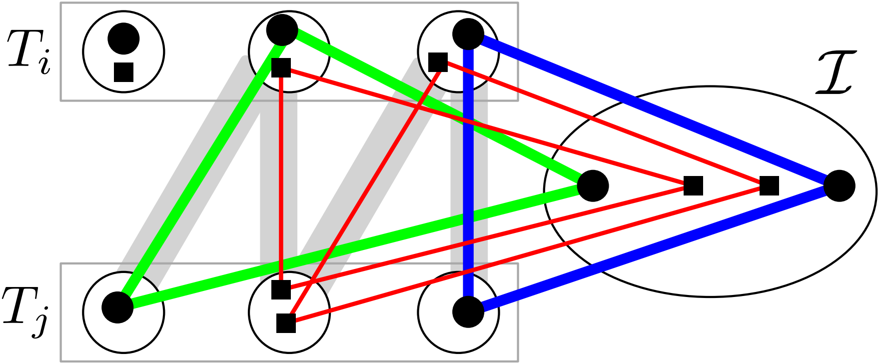


- Few edges inside $\mathcal{I}$ and few pairs in $\mathcal{I}$ make edges with vertices in two color classes of many tripartite graphs.
- $\delta_1(H)$ implies that on average the link graph of a pair of tripartite graphs has 5 edges.
- Suppose for many pairs the link graph has perfect matching.

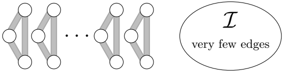




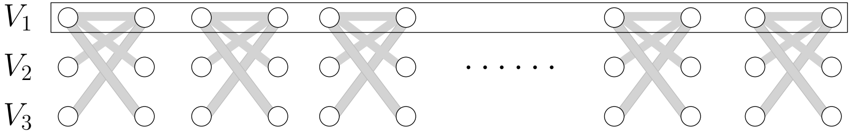
- Suppose for many pairs the link graph has a B_{320}.



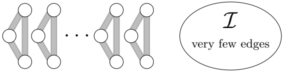
3-graphs - vertex degree: Almost perfect cover



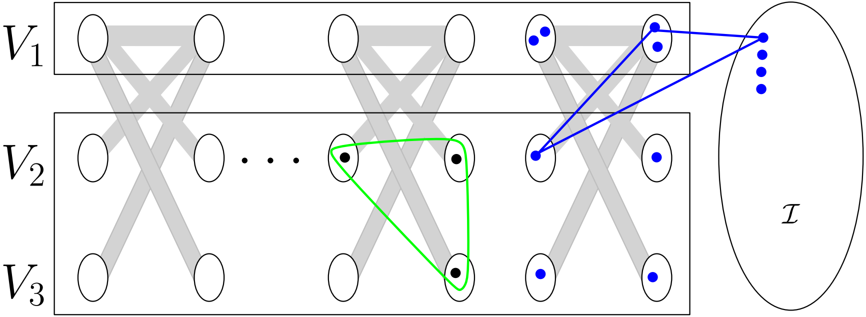
- Few edges inside $\mathcal{I}$.
- Few pairs in $\mathcal{I}$ make edges with vertices in two color classes of many tripartite graphs.
- $\delta_1(H)$ implies that on average the link graph of a pair of tripartite graphs has 5 edges.
- For few pairs the link graph has perfect matching or has a B_{320}.



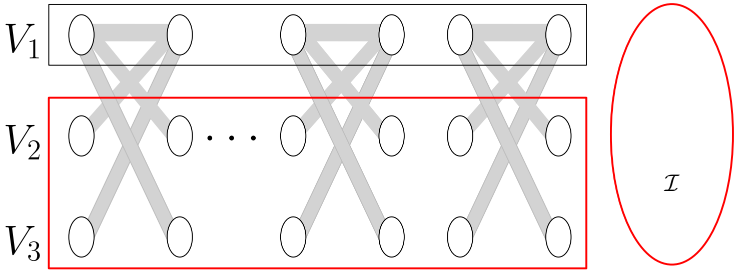
3-graphs - vertex degree: Almost perfect cover



- For almost all pairs of tripartite graphs, the link graph is ismorphic to B_{311}.



- Few edges in $V_2 \cup V_3$
- Similarly few edges with two vertices in $\mathcal{I}$ and one in $V_2 \cup V_3$.



- Few edges in $\mathcal{I}$
- Few edges in $V_2 \cup V_3$
- Few edges with two vertices in $\mathcal{I}$ and one in $V_2 \cup V_3$.
- By definition of B_{311}, few edges with one vertex in $\mathcal{I}$ and two in $V_2 \cup V_3$.
- So few edges in $V_2 \cup V_3 \cup \mathcal{I}$, while its size is $\sim 2n/3$, hence H is close to the extremal construction.

Thank You!