Eulerain and Stirling numbers over multisets

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POSTECH
Eulerian numbers I

Let $\left\langle \frac{d}{i} \right\rangle$ be an **Eulerian number**, which is the number of permutations of $\{1, 2, \ldots, d\}$ with $i$ descents.

**Example**

Let $\sigma = [1, 3, 2, 5, 4]$ be a permutation of $\{1, 2, 3, 4, 5\}$. The number of descent in $\sigma$ is 2, that is, $3 > 2$ and $5 > 4$. 
Eulerian numbers II

Worpitzky identity

\[ x^d = \sum_{i=0}^{d-1} \binom{d}{i} (x - 1 - i + d), \]

Carlitz identity (A \( q \)-analog of Worpitzky identity)

\[
\left[ \begin{array}{c} x \\ 1 \end{array} \right]_q^d = \sum_{i=1}^{d} A_{d,i}(q) \left[ \begin{array}{c} x - 1 + i \\ d \end{array} \right]_q
\]

where \( \left[ \begin{array}{c} x \\ m \end{array} \right]_q = \prod_{i=1}^{m}(1 - q^{x-m+i})/(1 - q^i) \) and \( A_{d,i}(q) \) is a polynomial of \( q \).
**Observation**

- Ordinary sets \( \{1, 2, \ldots, l\} \rightarrow \text{multisets} \{1^{d_1}, 2^{d_2}, \ldots, l^{d_l}\} \).
- Eulerian numbers \( \rightarrow \text{multiset Eulerian numbers} \).
- Worpitzky identity \( \rightarrow \text{a multiset version} \).
- Carlitz identity \( \rightarrow \text{a multiset version} \).
Stirling numbers of the second kind

Let \( \{\binom{d}{k}\} \) be a \textit{Stirling number of the second kind}, which is the number of partitions of a \( d \) element set into \( k \) nonempty sets.

\[
xd = \sum_{k=1}^{d} k! \binom{d}{k} \binom{x}{k}
\]

We call \( k! \binom{d}{k} \) an \textit{ordered Stirling numbers of the second kind}.
Observation

- Ordered Stirling numbers $\rightarrow$ multiset ordered Stirling numbers?
- Stirling identity $\rightarrow$ a multiset version? or a $q$-analog?
Goals

- Multiset versions of Worpitzky identity and Carlitz identity.
- A multiset version of Stirling identity and its $q$-analog.
- Computations of multiset Eulerian numbers and multiset ordered Stirling numbers of the second kind.
Basic ideas

- For a sequence of finite sets of lattice points $S_0$, $S_1$, $S_2$, ..., we compute the numbers of elements in $S_n$ by two different ways and obtain a polynomial identity.

- To obtain a $q$-analog of this identity, we compute the following generating function

$$\sum_{(x_1,x_2,...,x_d)\in S_n} q^{x_1+x_2+...+x_d}$$

by two different ways.
Notations I

- For a positive integer $n$, let $[n] = \{1, 2, \ldots, n\}$.

- We denote point in $\mathbb{R}^d$ by $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ and denote $q^\mathbf{x} = q^{x_1+x_2+\cdots+x_d}$.

- The zero vector of $\mathbb{R}^d$ is denoted by $\mathbf{e}_{d,0}$ and for each $1 \leq i \leq d$ the $i$th unit vector of $\mathbb{R}^d$ is denoted by $\mathbf{e}_{d,i} = (0, \ldots, 0, 1, 0, \ldots, 0)$ (with 1 in the $i$th coordinate).
Notations II

- We define $\alpha_d^0$ to be a $d$-dimensional simplex whose vertexes are $\mathbf{e}_d, 0, \mathbf{e}_d, 0 + \mathbf{e}_d, 1, \ldots, \mathbf{e}_d, 0 + \mathbf{e}_d, 1 + \cdots + \mathbf{e}_d, d$. Note that $\alpha_0^d$ is the set of points $x$ such that $1 \geq x_1 \geq x_2 \geq \cdots \geq x_d \geq 0$.

- Let $S(d) = \{1^{d_1}, 2^{d_2}, \ldots, l^{d_l}\}$ be a multiset such that for each $1 \leq j \leq l$ the number of $j$ in $S(d)$ is $d_j$. We define $\mathcal{S}(d)$ to be the permutation set of $S(d)$ and denote a permutation of $S(d)$ by $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_d]$. 


The product of simplexes I

- Let $d = d_1 + d_2 + \cdots + d_l$ be a sum of nonnegative integers. For each $1 \leq j \leq l$, writing a point of $\mathbb{R}^{d_j}$ by $x_j = (x_{j,1}, x_{j,2}, \ldots, x_{j,d_j})$, we also denote a point of $\mathbb{R}^d = \prod_{j=1}^l \mathbb{R}^{d_j}$ by $x = \prod_{j=1}^l x_j$.

- Let $d = (d_1, d_2, \ldots, d_l)$. We define $\alpha^d = \prod_{j=1}^l \alpha_0^{d_j}$ and denote the vertexes of $\alpha^d$ by $e_{i_1,i_2,\ldots,i_l} = \prod_{j=1}^l (e_{d_j,0} + e_{d_j,1} + \cdots + e_{d_j,i_j})$ for $(i_1, i_2, \ldots, i_l) \in \prod_{j=1}^l \{0, 1, \ldots, d_j\}$. 

For a permutation $\sigma$ of $S(d)$, we define $\alpha^d(\sigma)$ to be a $d$-simplex with the vertexes $e_{i_0,1,i_0,2,\ldots,i_0,l}$ for $0 \leq h \leq d$ such that

$$
\begin{cases}
(i_{0,1}, i_{0,2}, \ldots, i_{0,l}) = e_{l,0}, \\
(i_{h,1}, i_{h,2}, \ldots, i_{h,l}) = \sum_{a=1}^{h} e_{l,\sigma_a} \text{ for } 1 \leq h \leq d.
\end{cases}
$$
We can easily show that

$$\alpha^d = \bigcup_{\sigma \in S(d)} \alpha^d(\sigma).$$

Moreover, the set $T_{\alpha^d}$ which is composed of the $d$-simplexes $\alpha^d(\sigma)$ for $\sigma \in S(d)$ and their faces is a triangulation of $\alpha^d$. 
**Main idea**

- Let $R$ be a subset of $\mathbb{R}^d$ and $n$ be a nonnegative integer. We denote $nR = \{nr | r \in R\}$ and $\mathbb{Z}(R) = \{x \in R | x \in \mathbb{N}^d\}$.

- To obtain a multiset version of Worpitzky identity, we will compute $|\mathbb{Z}(n\alpha^d)|$ by two different ways.

- To obtain a multiset version of Carlitz identity, we will compute $f_1(q) = \sum_{x \in \mathbb{Z}(n\alpha^d)} q^x$ by two different ways.
The first decomposition of $\alpha^d$

For a permutation $\sigma$ of $S(d)$, let $D(\sigma)$ be the descent set of $\sigma$, that is, the set of indexes $h$ such that $\sigma_h > \sigma_{h+1}$. We define

$$A(\sigma) = \{ x \in \alpha^d(\sigma) \mid x_{\sigma_h,i_h} > x_{\sigma_{h+1},i_{h+1}} \text{ for } h \in D(\sigma) \}.$$

We define the first decomposition of $\alpha^d$ to be

$$\alpha^d = \biguplus_{\sigma \in S(d)} A(\sigma),$$

where $\biguplus$ is the disjoint union.
A multiset version of Worpitzky identity I

By definition,

\[ |\mathbb{Z}(n\alpha^d)| = \prod_{j=1}^{l} |\mathbb{Z}(n\alpha^{d_j})| = \prod_{j=1}^{l} \binom{n + d_j}{d_j}. \]

\[ x \in \mathbb{Z}(nA(\sigma)) \text{ if and only if } x \text{ is a lattice point such that} \]

1. \( n \geq x_{\sigma_1,i_1} \geq x_{\sigma_2,i_2} \geq \cdots \geq x_{\sigma_d,i_d} \geq 0, \)
2. \( x_{\sigma_h,i_h} \geq x_{\sigma_{h+1},i_{h+1}} + 1 \) for \( h \in D(\sigma). \)

Thus

\[ |\mathbb{Z}(nA(\sigma))| = \binom{n - |D(\sigma)| + d}{d}. \]
Therefore if we denote by $\binom{d}{i}$ the number of permutations of $S(d)$ with $i$ descents, called a *multiset Eulerian number*, then by the first decomposition of $\alpha^d$ we obtain

$$\begin{align*}
|\mathbb{Z}(n\alpha^d)| &= |\mathbb{Z}( \bigcup_{\sigma \in \mathcal{S}(d)} nA(\sigma))| = \sum_{\sigma \in \mathcal{S}(d)} |\mathbb{Z}(nA(\sigma))|
\end{align*}$$

$$= \sum_{\sigma \in \mathcal{S}(d)} \binom{n - |D(\sigma)| + d}{d} = \sum_{i=1}^{d-1} \binom{d}{i} \binom{n - i + d}{d}.$$
A multiset version of Worpitzky identity III

As a result,

$$\prod_{j=1}^{l} \binom{n + d_j}{d_j} = \sum_{i=0}^{d-1} \left< \begin{array}{c} d \\ i \end{array} \right> \binom{n - i + d}{d}.$$ 

A multiset version of Worpitzky identity

$$\prod_{j=1}^{l} \binom{x + d_j}{d_j} = \sum_{i=0}^{d-1} \left< \begin{array}{c} d \\ i \end{array} \right> \binom{x - i + d}{d}.$$
Computations of multiset Eulerian numbers I

- From identity (3), we can obtain the following matrix identity

\[
\left[ \begin{array}{c}
\prod_{j=1}^{l} \binom{d_j}{d_j} \\
\prod_{j=1}^{l} \binom{1+d_j}{d_j} \\
\vdots \\
\prod_{j=1}^{l} \binom{d-1+d_j}{d_j}
\end{array} \right] = 
\left[ \begin{array}{cccc}
\binom{d}{d} & 0 & \cdots & 0 \\
\binom{d+1}{d} & \binom{d}{d} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\binom{2d-1}{d} & \binom{2d-2}{d} & \cdots & \binom{d}{d}
\end{array} \right] 
\left[ \begin{array}{c}
\langle d \rangle \\
\langle d \rangle \\
\vdots \\
\langle d \rangle \\
\langle d \rangle
\end{array} \right].
\]

- By using the Gaussian elimination, we obtain

\[
\langle d \rangle_i = \sum_{h=0}^{i} (-1)^{i-h} \binom{d+1}{i-h} \prod_{j=1}^{l} \binom{h+d_j}{d_j}.
\]
A multiset version of Carlitz identity I

- \( x_j \in \mathbb{Z}(n\alpha_0^{d_j}) \) if and only if \( n \geq x_{j,1} \geq x_{j,2} \geq \cdots \geq x_{j,d_j} \geq 0 \).

Thus

\[
\sum_{x_j \in \mathbb{Z}(n\alpha_0^{d_j})} q^{x_j} = \begin{bmatrix} n + d_j \\ d_j \end{bmatrix}_q.
\]

- From this result, we obtain

\[
f_1(q) = \prod_{j=1}^l \sum_{x_j \in \mathbb{Z}(n\alpha_0^{d_j})} q^{x_j} = \prod_{j=1}^l \begin{bmatrix} n + d_j \\ d_j \end{bmatrix}_q.
\]
A multiset version of Carlitz identity II

For a permutation $\sigma$ of $S(d)$, we define the major index of $\sigma$ to be $maj(\sigma) = \sum_{j \in D(\sigma)} j$. A point $x$ is in $\mathbb{Z}(nA(\sigma))$ if and only if

$$\begin{align*}
&\begin{cases}
x_{\sigma_h,i_h} \geq x_{\sigma_{h+1},i_{h+1}} & \text{for } h \in [d + 1] \setminus D(\sigma) \\
x_{\sigma_h,i_h} \geq x_{\sigma_{h+1},i_{h+1}} + 1 & \text{for } h \in D(\sigma)
\end{cases}
\end{align*}$$

Therefore

$$\sum_{x \in \mathbb{Z}(nA(\sigma))} q^x = q^{maj(\sigma)} \left[ n - |D(\sigma)| + d \atop d \right]_q.$$
A multiset version of Carlitz identity III

Since $\mathbb{Z}(n\alpha^d) = \bigcup_{\sigma \in \mathcal{S}(d)} \mathbb{Z}(nA(\sigma))$, we obtain

$$f_1(q) = \sum_{\sigma \in \mathcal{S}(d)} \sum_{x \in \mathbb{Z}(nA(\sigma))} q^x$$

$$= \sum_{\sigma \in \mathcal{S}(d)} q^{maj(\sigma)} \left[ n - |D(\sigma)| + d \right]_q$$

$$= \sum_{i=1}^{d} A_{d,i}(q) \left[ n + i \right]_q$$

where $A_{d,i}(q) = \sum_{|D(\sigma)| = d-i} q^{maj(\sigma)}$. 
A multiset version of Carlitz identity IV

As a result,

\[
\prod_{j=1}^{l} \left[ \begin{array}{c} n + d_j \\ d_j \end{array} \right]_q = \sum_{i=1}^{d} A_{d,i}(q) \left[ \begin{array}{c} n + i \\ d \end{array} \right]_q.
\]
The second decomposition of $\alpha^d$ I

- For two vectors $\mathbf{v} = (v_1, v_2, \ldots, v_l)$ and $\mathbf{v'} = (v'_1, v'_2, \ldots, v'_l)$, we define $\mathbf{v} \leq \mathbf{v'}$ if $v_i \leq v'_i$ for all $1 \leq i \leq l$. Let $\mathcal{I}(\alpha^d)$ be the set of simplexes in $\mathcal{T}_{\alpha^d}$ that contains $e_{0,0,\ldots,0}$ and $e_{d_1,d_2,\ldots,d_l}$.

- Let $\alpha^k$ be a $k$-dimensional simplex in $\mathcal{I}(\alpha^d)$. Then the vertex set of $\alpha^k$ is of the form $\{e_{i_h,1,i_h,2,\ldots,i_h,l} \mid 0 \leq h \leq k\}$ where

$$
\begin{cases}
(i_{0,1}, i_{0,2}, \ldots, i_{0,l}) = 0 \\
(i_{h,1}, i_{h,2}, \ldots, i_{h,l}) < (i_{h+1,1}, i_{h+1,2}, \ldots, i_{h+1,l}) \text{ for } 0 \leq h \leq k - 1 \\
(i_{k,1}, i_{k,2}, \ldots, i_{k,l}) = (d_1, d_2, \ldots, d_l).
\end{cases}
$$

(1)
The second decomposition of $\alpha^d$ II

- Let $I(\alpha^k)$ be a subset of $\alpha^k$ which is composed of the set of convex sums $x = \sum_{h=0}^{k} c_h e_{i_h,1,i_h,2,\ldots,i_h,l}$ such that $c_h > 0$ for $1 \leq h \leq k - 1$.

- We define the *second decomposition* of $\alpha^d$ to be

$$
\alpha^d = \bigcup_{k=1}^{d} \bigcup_{\alpha^k \in I(\alpha^d)} I(\alpha^k).
$$
A multiset version of Stirling identity I

Let $\alpha^k$ be a $k$-dimensional simplex in $\mathcal{I}(\alpha^d)$ with the vertex set $\{e_{i_h,1, i_h,2, \ldots, i_h,l} \mid 0 \leq h \leq k\}$. For each $1 \leq h \leq k$ if we denote $S_h = S(i_{h,1}, i_{h,2}, \ldots, i_{h,l}) \setminus S(i_{h-1,1}, i_{h-1,2}, \ldots, i_{h-1,l})$, then $(S_1, S_2, \ldots, S_k)$ is an ordered partition of $S(d)$ into $k$ nonempty multisets.
Therefore the number of $k$-dimensional simplexes in $\mathcal{I}(\alpha^d)$ is the number of ordered partitions of $S(d)$ into $k$ nonempty multisets. We call this number an ordered multiset Stirling number of the second kind and denote it by $\{d\}_O^k$. Note that if $d_1 = d_2 = \cdots = d_l = 1$, then $\{d\}_O^k = k!\{l\}_k$. 
The number of elements in the set $\mathbb{Z}(nl(\alpha^k))$ is $\binom{n+1}{k}$. Thus by the second decomposition of $\alpha^d$ it follows that

$$|\mathbb{Z}(n\alpha^d)| = |\mathbb{Z}(n \bigcup_{k=1}^{d} \bigcup_{\alpha^k \in \mathcal{I}(\alpha^{d_1, \ldots, d_l})} l(\alpha^k))|$$

$$= \sum_{k=1}^{d} \sum_{\alpha^k \in \mathcal{I}(\alpha^d)} |\mathbb{Z}(nl(\alpha^k))|$$

$$= \sum_{k=1}^{d} \left\{ \begin{array}{c} d \\ k \end{array} \right\} O(\binom{n+1}{k}).$$
A multiset version of Stirling identity IV

As a result,

$$\prod_{j=1}^{l} \binom{n + d_j}{d_j} = \sum_{k=1}^{d} \{d\} \binom{n + 1}{k}.$$

A multiset version of Stirling identity

$$\prod_{j=1}^{l} \binom{x + d_j}{d_j} = \sum_{k=1}^{d} \{d\} \binom{x + 1}{k}.$$
Introduction

A triangulation of the product of simplexes

Multiset Eulerian numbers

Multiset ordered Stirling numbers of the second kind

Computations of multiset Stirling numbers of the second kind I

- Multiset ordered Stirling numbers of the second kind satisfy

\[
\begin{pmatrix}
\Pi_{j=1}^l \binom{d_j}{d_j} \\
\Pi_{j=1}^l \binom{1+d_j}{d_j} \\
\ldots \\
\Pi_{j=1}^l \binom{d-1+d_j}{d_j}
\end{pmatrix}
= 
\begin{pmatrix}
\binom{1}{1} & 0 & \ldots & 0 \\
\binom{2}{1} & \binom{2}{2} & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
\binom{d}{1} & \binom{d}{2} & \ldots & \binom{d}{d}
\end{pmatrix}
\begin{pmatrix}
\{d\}^1\_O \\
\{d\}^2\_O \\
\ldots \\
\{d\}^d\_O
\end{pmatrix}.
\]
By the Gaussian elimination, we obtain

\[
\left\{ \begin{array}{c} d \\ k \end{array} \right\}_O = \sum_{h=0}^{k-1} (-1)^{k-1-h} \binom{k}{h} \prod_{j=1}^{l} \binom{h + d_j}{d_j}.
\]
Let $\alpha^k$ be a $k$-dimensional simplex in $\mathcal{I}(\alpha^d)$ with the vertex set $\{e_{i_{h,1},i_{h,2},...,i_{h,l}} | 0 \leq h \leq k\}$. We define the \textit{major index} of $\alpha^k$ to be

$$maj(\alpha^k) = \sum_{h=1}^{k-1} \sum_{j=1}^{l} i_{h,j}.$$
A $q$-analog of a multiset version of Stirling identity II

- Each $\alpha^k$ in $I(\alpha^d)$ satisfies $\sum_{x \in \mathbb{Z}(nl(\alpha^k))} q^x = q^{maj(\alpha^k)} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q$.

Thus it follows that

$$f_1(q) = \sum_{k=1}^{d} \sum \sum q^x$$

$$= \sum_{k=1}^{d} q^{maj(\alpha^k)} \begin{bmatrix} n + 1 \\ k \end{bmatrix}_q$$

$$= \sum_{k=1}^{d} B_{d,k}(q) \begin{bmatrix} n + 1 \\ k \end{bmatrix}_q$$

where $B_{d,k}(q) = \sum_{\alpha^k \in I(\alpha^d)} q^{maj(\alpha^k)}$. 
A $q$-analog of a multiset version of Stirling identity III

As a result,

$$\prod_{j=1}^{l} \left[ \begin{array}{c} n + d_j \\ d_j \end{array} \right]_q = \sum_{k=1}^{d} B_{d,k}(q) \left[ \begin{array}{c} n + 1 \\ k \end{array} \right]_q.$$