Perfect 1-Factorisations of Circulant Graphs of Degree 4

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OUTLINE

- definitions and history
- what does bipartite have to do with it?
- our results
- future research
Basic Definitions

A 1-factor of a graph $G$ is a spanning 1-regular subgraph of $G$.

A 1-factorisation of a graph $G$ is a partition of the edges of $G$ into 1-factors.

A 1-factorisation is perfect if the union of every pair of distinct 1-factors forms a Hamilton cycle.

The above 1-factorisation is not a P1F.
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Consider $K_6$: 

![Graph of $K_6$]
Example:

Consider $K_6$: 

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P1Fs of Circulants 
June 2013 
4 / 18
Example:

Consider $K_6$: 

![Graph Image]
Example:

Consider $K_6$: 

![Graph](image)
Consider $K_6$: 
EXAMPLE:

Consider $K_6$:
Conjecture (Kotzig, ’64)

The complete graph $K_{2n}$ admits a P1F for all $n \geq 2$. 
Conjecture for Complete Graphs

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Conjecture for Complete Graphs

Conjecture (Kotzig, ’64)

The complete graph $K_{2n}$ admits a P1F for all $n \geq 2$.

- proven when $n$ is an odd prime
- proven when $2n - 1$ is an odd prime
- small values (upto $K_{52}$) and other sporadic values
Suppose \( n \) is even and \( S \subseteq \{1, 2, \ldots, \frac{n}{2}\} \).

The circulant graph on \( n \) vertices with connection set \( S \), denoted \( \text{Circ}(n, S) \), has vertex set \( V = \{0, 1, \ldots, n - 1\} \) and edge set 
\[
E = \{\{x, x + s \pmod{n}\} \mid x \in V, s \in S\}. 
\]

Example: \( \text{Circ}(10, \{1, 2, 5\}) \)
Suppose $n$ is even and $S \subseteq \{1, 2, \ldots, \frac{n}{2}\}$.

The circulant graph on $n$ vertices with connection set $S$, denoted $\text{Circ}(n, S)$, has vertex set $V = \{0, 1, \ldots, n - 1\}$ and edge set $E = \{\{x, x + s \pmod{n}\} \mid x \in V, s \in S\}$.

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Example: $\text{Circ}(10, \{1, 2, 5\})$

A 4-regular circulant has $S = \{a, b\}$ where $1 \leq a < b < \frac{n}{2}$. 
Theorem (Stong, ’85)

A connected Cayley graph on a finite Abelian group of even order has a 1-factorisation.
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**Theorem (Bermond, Favaron, Maheo, ’89)**

A 4-regular connected Cayley graph on a finite Abelian group can be decomposed into two Hamilton cycles.
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**Problem**

Characterise the circulant graphs that admit a P1F.
Bipartite Case

Theorem (Kotzig, ’64)

If $G$ is a bipartite $r$-regular graph with $r > 2$ and $G$ admits a P1F, then $|V(G)| \equiv 2 \pmod{4}$.

Proof (idea): Suppose $|V(G)| = 2n$ where $n$ is even and there is a P1F $F_1, F_2, \ldots, F_r$. Example: $n = 4$

$\sigma - 1_j \sigma_i$ is an odd permutation $\Rightarrow \sigma_i, \sigma_j$ have different parities

This holds for all pairs $i, j$ $\Rightarrow r \leq 2$ ($\Rightarrow \Leftarrow$) $\square$
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\[ a_1 \rightarrow b_1 \]
\[ a_2 \rightarrow b_2 \]
\[ a_3 \rightarrow b_3 \]
\[ a_4 \rightarrow b_4 \]
**Theorem (Kotzig, ’64)**

*If G is a bipartite r-regular graph with r > 2 and G admits a P1F, then \(|V(G)| \equiv 2 \pmod{4}\).*

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![Diagram](image-url)
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$$\sigma_1 = (13)$$
**Theorem (Kotzig, ’64)**

If \( G \) is a bipartite \( r \)-regular graph with \( r > 2 \) and \( G \) admits a P1F, then \( |V(G)| \equiv 2 \pmod{4} \).

Proof (idea): Suppose \( |V(G)| = 2n \) where \( n \) is even and there is a P1F \( F_1, F_2, \ldots, F_r \). Example: \( n = 4 \)

\[
\begin{align*}
F_1 & : a_1 \rightarrow b_1 \\
F_2 & : a_2 \rightarrow b_2
\end{align*}
\]

\[
\begin{align*}
a_3 & \rightarrow b_3 \\
a_4 & \rightarrow b_4
\end{align*}
\]

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**Bipartite Case**

**Theorem (Kotzig, ’64)**

*If G is a bipartite r-regular graph with r > 2 and G admits a P1F, then |V(G)| ≡ 2 (mod 4).*

Proof (idea): Suppose |V(G)| = 2n where n is even and there is a P1F $F_1, F_2, \ldots, F_r$. Example: $n = 4$

\[
F_1 \quad a_1 \rightarrow b_1 \quad \sigma_1 = (13)
\]

\[
F_2 \quad a_2 \rightarrow b_2 \quad \sigma_2 = (243)
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- $F_1$  
  - $a_1 \rightarrow b_1$
  - $\sigma_1 = (13)$

- $F_2$  
  - $a_2 \rightarrow b_2$
  - $\sigma_2 = (243)$

- $F_1 \cup F_2$  
  - $a_3 \rightarrow b_3$
  - $1 \rightarrow 3$ by $\sigma_1$
  - $3 \rightarrow 4$ by $\sigma_2^{-1}$
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$$a_1 \rightarrow b_1$$

$$F_2$$

$$a_2 \rightarrow b_2$$

$$F_1 \cup F_2$$

$$a_3 \rightarrow b_3$$

$$a_4 \rightarrow b_4$$

1 $\rightarrow$ 3 by $\sigma_1^{-1}$

3 $\rightarrow$ 4 by $\sigma_2^{-1}$

$1 \rightarrow 4$ by $\sigma_1, \sigma_2$
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```
F_1
a_1 - b_1

F_2
a_2 - b_2

F_1 \cup F_2
a_3 - b_3
1 \rightarrow 3 \text{ by } \sigma_1
3 \rightarrow 4 \text{ by } \sigma_2^{-1}

\sigma_1 = (13)
\sigma_2 = (243)
\sigma_{1,2} = \sigma_2^{-1} \sigma_1
= (234)(13)
= (1423)
1 \rightarrow 4 \text{ by } \sigma_{1,2}
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- $F_1$
- $F_2$
- $F_1 \cup F_2$

Hamilton cycle

- $a_1 \rightarrow b_1$
- $a_2 \rightarrow b_2$
- $a_3 \rightarrow b_3$
- $a_4 \rightarrow b_4$

$\sigma_1 = (13)$

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$= (234)(13)$

$= (1423)$

single $n$-cycle
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![Graph with vertices and edges illustrating the theorem's conditions](image)

- $F_1$
- $F_2$
- $F_1 \cup F_2$
- $a_1$, $a_2$, $a_3$, $a_4$
- $b_1$, $b_2$, $b_3$, $b_4$

- $\sigma_1 = (13)$
- $\sigma_2 = (243)$
- $\sigma_{1,2} = \sigma_2^{-1}\sigma_1$
  - $= (234)(13)$
  - $= (1423)$

Hamilton cycle: single $n$-cycle

$\sigma_j^{-1}\sigma_i$ is an odd permutation $\Rightarrow \sigma_i, \sigma_j$ have different parities
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& & \quad = (1423)
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Hamilton cycle  \quad single \ n\text{-cycle}

$\sigma_j^{-1}\sigma_i$ is an odd permutation $\Rightarrow \sigma_i, \sigma_j$ have different parities

This holds for all pairs $i, j \Rightarrow r \leq 2$

$(\Rightarrow \Leftarrow)$

\[\square\]
Bipartite Case

\( \text{Circ}(n, \{a, b\}) \) is bipartite \iff \ a, b \) are both odd.
**Bipartite Case**

\[ \text{Circ}(n, \{a, b\}) \text{ is bipartite } \iff a, b \text{ are both odd.} \]

**Corollary**

_If a, b are both odd and \( \text{Circ}(n, \{a, b\}) \) admits a P1F, then \( n \equiv 2 \pmod{4} \)._
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Is this necessary condition sufficient?


**Bipartite Case**

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*If \( a, b \) are both odd and \( \text{Circ}(n, \{a, b\}) \) admits a P1F, then \( n \equiv 2 \pmod{4} \).*

*Is this necessary condition sufficient?*

**Theorem (S.H. and Maenhaut)**

*If \( n > 6 \), then a connected 3-regular circulant graph \( G \) of order \( n \) admits a P1F if and only if \( n \equiv 2 \pmod{4} \) and \( G \) is bipartite.*
Fact (computer results)

For $8 \leq n \leq 28$, a connected 4-regular circulant $G = \text{Circ}(n, \{a, b\})$ has a P1F if and only if $n \equiv 2 \pmod{4}$ and $G$ is bipartite.
Our Results

Fact (computer results)

For $8 \leq n \leq 28$, a connected 4-regular circulant $G = \text{Circ}(n, \{a, b\})$ has a P1F if and only if $n \equiv 2 \pmod{4}$ and $G$ is bipartite.

Problem 1: Show that a non-bipartite 4-regular circulant of order $\geq 8$ does not admit a P1F.

Problem 2: Construct P1Fs for families of bipartite 4-regular circulants of order $2 \pmod{4}$.

Not So Fast...

$\text{Circ}(30, \{1, 11\})$ does NOT admit a P1F

Problem 3: Why is there no P1F of $\text{Circ}(30, \{1, 11\})$? Are there others like it?
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Theorem (S.H. and Maenhaut)

If $n > 6$ is even, then any connected 4-regular circulant graph isomorphic to $\text{Circ}(n, \{1, 2\})$ or to $\text{Circ}(n, \{1, 4\})$ does not admit a P1F.
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If $n > 6$ is even, then any connected 4-regular circulant graph isomorphic to $\text{Circ}(n, \{1, 2\})$ or to $\text{Circ}(n, \{1, 4\})$ does not admit a P1F.

**Theorem (S.H. and Maenhaut)**

Suppose $n > 6$ and $n \equiv 2 \pmod{4}$.

Then $\text{Circ}(n, \{1, \frac{n}{2} - 1\})$ does not admit a P1F.
Problem 2: Construct P1Fs of bipartite 4-regular circulants of order 2 (mod 4).
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Theorem (S.H. and Maenhaut)

For \( n > 6 \), \( \text{Circ}(n, \{1, 3\}) \) admits a P1F \( \iff n \equiv 2 \pmod{4} \).
**Problem 2:** Construct P1Fs of bipartite 4-regular circulants of order $2 \pmod{4}$.

**Theorem (S.H. and Maenhaut)**

For $n > 6$, $\text{Circ}(n, \{1, 3\})$ admits a P1F $\iff n \equiv 2 \pmod{4}$.

**Theorem (S.H. and Maenhaut)**

Suppose $n \geq 14$, $n \equiv 2 \pmod{4}$ and $5 \leq b \leq \frac{n}{2} - 2$ is an odd integer. If $\gcd(n, b) = 1$ and $\gcd(n, b - 1) = \gcd(n, b + 1) = 2$ then any circulant isomorphic to $\text{Circ}(n, \{1, b\})$ admits a P1F.
**Bipartite Constructions**

**Problem 2:** Construct P1Fs of bipartite 4-regular circulants of order $2 \pmod{4}$.

**Theorem (S.H. and Maenhaut)**

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Suppose $n \geq 14$ and $n \equiv 2 \pmod{4}$. Then any circulant isomorphic to $\text{Circ}(n, \{1, \frac{n}{2} - 2\})$ admits a P1F.
## Bipartite Constructions

<table>
<thead>
<tr>
<th>$n$</th>
<th>P1F</th>
<th>unknown</th>
<th>no P1F</th>
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<tr>
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<td>none</td>
<td>${1,11}$</td>
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<tr>
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<tr>
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<td>${1,3}$ ${1,5}$ ${1,7}$ ${1,9}$</td>
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<td>none</td>
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<tr>
<td>42</td>
<td>${1,3}$ ${1,5}$ ${1,11}$ ${1,13}$</td>
<td>${1,7}$ ${1,9}$ ${1,15}$ ${3,7}$</td>
<td></td>
</tr>
<tr>
<td>46</td>
<td>${1,3}$ ${1,5}$ ${1,7}$ ${1,11}$ ${1,17}$</td>
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<td>none</td>
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<td>${1,3}$ ${1,7}$ ${1,9}$ ${1,13}$</td>
<td>${1,5}$ ${1,15}$ ${1,19}$</td>
<td></td>
</tr>
</tbody>
</table>

- from $\{1,3\}$ result
- from $\{1,b\}$ result
- from $\{1, \frac{n}{2} - 2\}$ result
- from other existence results
What about $\text{Circ}(30, \{1, 11\})$?

**Problem 3:** Why is there no P1F of $\text{Circ}(30, \{1, 11\})$? Are there others like it?
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**Problem 3:** Why is there no P1F of \text{Circ}(30, \{1, 11\})? Are there others like it?

\text{Circ}(30, \{1, 11\}) can be drawn another way...
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**Problem 3:** Why is there no P1F of $\text{Circ}(30, \{1, 11\})$? Are there others like it?

$\text{Circ}(30, \{1, 11\})$ can be drawn another way...
**Implications**

**Theorem (S.H. and Maenhaut)**

Suppose \( k \equiv 2 \pmod{4} \) and \( k > 6 \).

If \( k \equiv 10 \pmod{12} \) then \( \text{Circ}(3k, \{1, k + 1\}) \) does not admit a P1F.
**Implications**

**Theorem (S.H. and Maenhaut)**

Suppose $k \equiv 2 \pmod{4}$ and $k > 6$.

If $k \equiv 10 \pmod{12}$ then $\text{Circ}(3k, \{1, k+1\})$ does not admit a P1F.

If $k \equiv 2 \pmod{12}$ then $\text{Circ}(3k, \{1, k-1\})$ admits a P1F.
**Theorem (S.H. and Maenhaut)**

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<table>
<thead>
<tr>
<th>( k \equiv 10 \pmod{12} )</th>
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Implications

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*If* \( k \equiv 10 \pmod{12} \) *then* \( \text{Circ}(3k, \{1, k + 1\}) \) **does not** admit a P1F.

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**Corollary (S.H. and Maenhaut)**

There is an infinite family of 4-regular bipartite circulant graphs of order \( n \equiv 2 \pmod{4} \) that do not admit a P1F.
Implications

By studying similar structures...
Implications

By studying similar structures...

**Theorem (S.H. and Maenhaut)**

*If* $k \equiv 22, 34, 46, 58 \pmod{60}$ *then there exists a P1F of* $\text{Circ}(5k, \{1, b\})$, *where* $b = k - 1, 2k + 1, 2k - 1, k + 1$, *respectively.*
Implications

By studying similar structures...

**Theorem (S.H. and Maenhaut)**

If $k \equiv 22, 34, 46, 58 \pmod{60}$ then there exists a P1F of $\text{Circ}(5k, \{1, b\})$, where $b = k - 1, 2k + 1, 2k - 1, k + 1$, respectively.

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<td>82</td>
<td>$\text{Circ}(410, {1, 81})$</td>
</tr>
<tr>
<td>34</td>
<td>$\text{Circ}(170, {1, 69})$</td>
<td>94</td>
<td>$\text{Circ}(470, {1, 189})$</td>
</tr>
<tr>
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<td>$\text{Circ}(230, {1, 91})$</td>
<td>106</td>
<td>$\text{Circ}(530, {1, 211})$</td>
</tr>
<tr>
<td>58</td>
<td>$\text{Circ}(290, {1, 59})$</td>
<td>118</td>
<td>$\text{Circ}(590, {1, 119})$</td>
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Future Research

Open Problem

Characterise the bipartite 4-regular circulants of order 2 (mod 4) that admit a P1F.
**Future Research**

**Open Problem**

Characterise the bipartite 4-regular circulants of order 2 (mod 4) that admit a P1F.

**Conjecture**

A non-bipartite 4-regular circulant of order at least 8 does not admit a P1F.
Thank you!

Any questions?