Forbidden Families of Configurations

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Consider the following family of subsets of \( \{1, 2, 3, 4\} \):
\[ \mathcal{A} = \{\emptyset, \{1, 2, 4\}, \{1, 4\}, \{1, 2\}, \{1, 2, 3\}, \{1, 3\}\} \]
The incidence matrix \( A \) of the family \( \mathcal{A} \) of subsets of \( \{1, 2, 3, 4\} \) is:

\[
A = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

**Definition** We say that a matrix \( A \) is *simple* if it is a \((0,1)\)-matrix with no repeated columns.

**Definition** We define \( \|A\| \) to be the number of columns in \( A \).
\[
\|A\| = 6 = |\mathcal{A}|
\]
**Definition** Given a matrix $F$, we say that $A$ has $F$ as a 
*configuration* (denoted $F \prec A$) if there is a submatrix of $A$ which is 
a row and column permutation of $F$.

\[
F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \prec A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}
\]
Definition Given a matrix $F$, we say that $A$ has $F$ as a\textit{ configuration} (denoted $F \prec A$) if there is a submatrix of $A$ which is a row and column permutation of $F$.

\[
F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \prec \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}
\]

Definitions

\[
\mathcal{F} = \{F_1, F_2, \ldots, F_t\}
\]

\[
\text{Avoid}(m, \mathcal{F}) = \{ A : A \text{ m-rowed simple, } F \not\prec A \text{ for all } F \in \mathcal{F} \}
\]

\[
\text{forb}(m, \mathcal{F}) = \max_A \{ \|A\| : A \in \text{Avoid}(m, \mathcal{F}) \}
\]
Definition Let $K_k$ be the $k \times 2^k$ simple matrix of all possible columns on $k$ rows.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$\text{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$

Theorem (Füredi 83). Let $F$ be a $k \times \ell$ matrix. Then

$$\text{forb}(m, F) = O(m^k).$$

Problem Given $\mathcal{F}$, can we predict the behaviour of $\text{forb}(m, \mathcal{F})$?
Balanced and Totally Balanced Matrices

Let $C_k$ denote the $k \times k$ vertex-edge incidence matrix of the cycle of length $k$.

e.g. $C_3 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $C_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.
Balanced and Totally Balanced Matrices

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Matrices in $\text{Avoid}(m, \{C_3, C_5, C_7, \ldots\})$ are called Balanced Matrices.

**Theorem** $\text{forb}(m, \{C_3, C_5, C_7, \ldots\}) = \text{forb}(m, C_3)$
Balanced and Totally Balanced Matrices

Let $C_k$ denote the $k \times k$ vertex-edge incidence matrix of the cycle of length $k$.

\[
\begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
\end{bmatrix}, \quad 
\begin{bmatrix}
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}.
\]

Matrices in Avoid($m, \{C_3, C_5, C_7, \ldots\}$) are called Balanced Matrices.

**Theorem** $\text{forb}(m, \{C_3, C_5, C_7, \ldots\}) = \text{forb}(m, C_3)$

Matrices in Avoid($m, \{C_3, C_4, C_5, C_6, \ldots\}$) are called Totally Balanced Matrices.

**Theorem** $\text{forb}(m, \{C_3, C_4, C_5, C_6, \ldots\}) = \text{forb}(m, C_3)$
**Remark** If $\mathcal{F} \subset \mathcal{F}$ then $\text{forb}(m, \mathcal{F}) \leq \text{forb}(m, \mathcal{F}')$.

The inequality $\text{forb}(m, \{C_3, C_4, C_5, C_6, \ldots\}) \leq \text{forb}(m, C_3)$ follows from the remark.

The equality follows from a result that any $m \times \text{forb}(m, C_3)$ simple matrix in Avoid($m, C_3$) is in fact totally balanced (A, 80).

Thus we conclude $\text{forb}(m, \{C_3, C_4, C_5, C_6, \ldots\}) = \text{forb}(m, C_3)$. 
A Product Construction

The building blocks of our product constructions are $I$, $I^c$ and $T$:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I^c_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
Definition  Given an $m_1 \times n_1$ matrix $A$ and a $m_2 \times n_2$ matrix $B$ we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of $A$ on top of a column of $B$. If $A$, $B$ are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Given $p$ simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.
**Definition**  Given an \( m_1 \times n_1 \) matrix \( A \) and a \( m_2 \times n_2 \) matrix \( B \) we define the product \( A \times B \) as the \( (m_1 + m_2) \times (n_1 n_2) \) matrix consisting of all \( n_1 n_2 \) possible columns formed from placing a column of \( A \) on top of a column of \( B \). If \( A, B \) are simple, then \( A \times B \) is simple. (A, Griggs, Sali 97)

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\times
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

Given \( p \) simple matrices \( A_1, A_2, \ldots, A_p \), each of size \( m/p \times m/p \), the \( p \)-fold product \( A_1 \times A_2 \times \cdots \times A_p \) is a simple matrix of size \( m \times (m^p/p^p) \) i.e. \( \Theta(m^p) \) columns.
**Definition** Let \( x(\mathcal{F}) \) denote the smallest \( p \) such that for every \( p \)-fold product \( A_1 \times A_2 \times \cdots \times A_p \), where each \( A_i \in \{ I_{m/p}, I_{m/p}^c, T_{m/p} \} \), there is some \( F \in \mathcal{F} \) with \( F \prec A_1 \times A_2 \times \cdots \times A_p \).

Thus there is some \((p - 1)\)-fold product \( A_1 \times A_2 \times \cdots \times A_{p-1} \in \text{Avoid}(m, \mathcal{F})\) showing that \( \text{forb}(m, \mathcal{F}) \) is \( \Omega(m^{p-1}) \).
The Conjecture

**Definition** Let $x(\mathcal{F})$ denote the smallest $p$ such that for every $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$, where each $A_i \in \{I_{m/p}, I_{m/p}^c, T_{m/p}\}$, there is some $F \in \mathcal{F}$ with $F \prec A_1 \times A_2 \times \cdots \times A_p$.

Thus there is some $(p-1)$-fold product $A_1 \times A_2 \times \cdots \times A_{p-1} \in \text{Avoid}(m, \mathcal{F})$ showing that $\text{forb}(m, \mathcal{F})$ is $\Omega(m^{p-1})$.

**Conjecture** (A, Sali 05) Let $|\mathcal{F}| = 1$. Then $\text{forb}(m, \mathcal{F})$ is $\Theta(m^{x(\mathcal{F})-1})$.

In other words, we predict our product constructions with the three building blocks $\{I, I^c, T\}$ determine the asymptotically best constructions when $|\mathcal{F}| = 1$. 

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Forbidden Families of Configurations
The Conjecture

**Definition** Let \( x(F) \) denote the smallest \( p \) such that for every \( p \)-fold product \( A_1 \times A_2 \times \cdots \times A_p \), where each \( A_i \in \{ I_{m/p}, I^c_{m/p}, T_{m/p} \} \), there is some \( F \in F \) with \( F \prec A_1 \times A_2 \times \cdots \times A_p \). Thus there is some \((p-1)\)-fold product \( A_1 \times A_2 \times \cdots \times A_{p-1} \in \text{Avoid}(m,F) \) showing that \( \text{forb}(m,F) \) is \( \Omega(m^{p-1}) \).

**Conjecture** (A, Sali 05) Let \( |F| = 1 \). Then \( \text{forb}(m,F) \) is \( \Theta(m^{x(F)-1}) \).

In other words, we predict our product constructions with the three building blocks \( \{ I, I^c, T \} \) determine the asymptotically best constructions when \( |F| = 1 \).

The conjecture has been verified for \( k \times \ell \ F \) where \( k = 2 \) (A, Griggs, Sali 97) and \( k = 3 \) (A, Sali 05) and \( \ell = 2 \) (A, Keevash 06).
**Definition** \( \text{ex}(m, H) \) is the maximum number of edges in a (simple) graph \( G \) on \( m \) vertices that has no subgraph \( H \).

\( A \in \text{Avoid}(m, \mathbf{1}_3) \) will be a matrix with up to \( m + 1 \) columns of sum 0 or sum 1 plus columns of sum 2 which can be viewed as the vertex-edge incidence matrix of a graph.

Let \( \text{Inc}(H) \) denote the \( |V(H)| \times |E(H)| \) vertex-edge incidence matrix associated with \( H \).

**Theorem** \( \text{forb}(m, \{\mathbf{1}_3, \text{Inc}(H)\}) = m + 1 + \text{ex}(m, H) \).
Definition \( \text{ex}(m, H) \) is the maximum number of edges in a (simple) graph \( G \) on \( m \) vertices that has no subgraph \( H \).

\( A \in \text{Avoid}(m, 1_3) \) will be a matrix with up to \( m + 1 \) columns of sum 0 or sum 1 plus columns of sum 2 which can be viewed as the vertex-edge incidence matrix of a graph.

Let \( \text{Inc}(H) \) denote the \( |V(H)| \times |E(H)| \) vertex-edge incidence matrix associated with \( H \).

**Theorem** \( \text{forb}(m, \{1_3, \text{Inc}(H)\}) = m + 1 + \text{ex}(m, H) \).

In this talk \( I(C_4) = C_4 \), \( I(C_6) = C_6 \).

**Theorem** \( \text{forb}(m, \{1_3, C_4\}) = m + 1 + \text{ex}(m, C_4) \) which is \( \Theta(m^{3/2}) \). note that \( x(\{1_3, C_4\}) = 2 \)

**Theorem** \( \text{forb}(m, \{1_3, C_6\}) = m + 1 + \text{ex}(m, C_6) \) which is \( \Theta(m^{4/3}) \). note that \( x(\{1_3, C_6\}) = 2 \)
Forbidden Families of Configurations

**Theorem** $forb(m, \{1_3, \text{Inc}(H)\}) = m + 1 + \text{ex}(m, H)$.

**Theorem** Let $T$ be a graph with no cycles. Then $\text{ex}(m, T)$ is $O(m)$.

**Corollary** Let $F$ be a (0,1)-matrix with column sums at most 2. Assume $C_k \not\preceq F$ for $k = 2, 3, \ldots$ (we don’t allow repeated columns of sum 2 but allow other repeated columns). Then $forb(m, \{1_3, F\})$ is $O(m)$.

**Proof:** We can find a graph $T$ with no cycles such that $F \prec \text{Inc}(T)$. Then $forb(m, \{1_3, F\}) \leq m + 1 + \text{ex}(m, T)$. 
**Theorem** (Balogh and Bollobás 05) Let $k$ be given. Then there is a constant $c_k$ so that $\text{forb}(m, \{I_k, I^c_k, T_k\}) = c_k$.

We note that $x(\{I_k, I^c_k, T_k\}) = 1$ and so there is no obvious product construction.

Note that $c_k \geq \binom{2k-2}{k-1}$ by taking all columns of column sum at most $k - 1$ that arise from the $k - 1$-fold product $T_{k-1} \times T_{k-1} \times \cdots \times T_{k-1}$.
Let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ and $\mathcal{G} = \{G_1, G_2, \ldots, G_\ell\}$.

**Lemma** Let $\mathcal{F}$ and $\mathcal{G}$ have the property that for every $G_i \in \mathcal{G}$, there is some $F_j \in \mathcal{F}$ with $F_j \prec G_i$. Then $\text{forb}(m, \mathcal{F}) \leq \text{forb}(m, \mathcal{G})$. 
Let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ and $\mathcal{G} = \{G_1, G_2, \ldots, G_\ell\}$.

**Lemma** Let $\mathcal{F}$ and $\mathcal{G}$ have the property that for every $G_i \in \mathcal{G}$, there is some $F_j \in \mathcal{F}$ with $F_j \prec G_i$. Then $\text{forb}(m, \mathcal{F}) \leq \text{forb}(m, \mathcal{G})$.

**Theorem** Let $\mathcal{F}$ be given. Then either $\text{forb}(m, \mathcal{F})$ is $O(1)$ or $\text{forb}(m, \mathcal{F})$ is $\Omega(m)$.
Let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ and $\mathcal{G} = \{G_1, G_2, \ldots, G_\ell\}$.

**Lemma** Let $\mathcal{F}$ and $\mathcal{G}$ have the property that for every $G_i \in \mathcal{G}$, there is some $F_j \in \mathcal{F}$ with $F_j \prec G_i$. Then $\text{forb}(m, \mathcal{F}) \leq \text{forb}(m, \mathcal{G})$.

**Theorem** Let $\mathcal{F}$ be given. Then either $\text{forb}(m, \mathcal{F})$ is $O(1)$ or $\text{forb}(m, \mathcal{F})$ is $\Omega(m)$.

**Proof:** We start using $\mathcal{G} = \{I_p, I^c_p, T_p\}$ with $p$ suitably large. Either we have the property that there is some $F_r \prec I_p$, and some $F_s \prec I^c_p$ and some $F_t \prec T_p$ in which case $\text{forb}(m, \mathcal{F}) \leq \text{forb}(m, \{I_p, I^c_p, T_p\})$ which is $O(1)$ or without loss of generality we have $F_j \not\prec I_p$ for all $j$ and hence $I_m \in \text{Avoid}(m, \mathcal{F})$ and so $\text{forb}(m, \mathcal{F})$ is $\Omega(m)$.
A pair of Configurations with quadratic bounds

e.g. $F_2(1, 2, 2, 1) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \not\preceq I \times I^c.$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{I_3} \times \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{I_3^c} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}_{I_m/2} \times I_{m/2}^c$$
A pair of Configurations with quadratic bounds

e.g. \( F_2(1, 2, 2, 1) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \not\prec I \times I^c. \)

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}_{I_3} \times \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}_{I_3^c} = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}
\]

\( I_{m/2} \times I_{m/2}^c \) is an \( m \times m^2/4 \) simple matrix avoiding \( F_2(1, 2, 2, 1) \), so \( \text{forb}(m, F_2(1, 2, 2, 1)) \) is \( \Omega(m^2) \).

(A, Ferguson, Sali 01 \( \text{forb}(m, F_2(1, 2, 2, 1)) = \left\lceil \frac{m^2}{4} \right\rceil + \binom{m}{1} + \binom{m}{0} \))
A pair of Configurations with quadratic bounds

e.g. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \not\preceq T \times T$. Also $I_3 \not\preceq I^c \times T$, $I_3 \not\preceq I^c \times I^c$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$
A pair of Configurations with quadratic bounds

e.g. \( I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \not\prec T \times T \). Also \( I_3 \not\prec I^c \times T, I_3 \not\prec I^c \times I^c \)

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

\( T_{m/2} \times T_{m/2} \) is an \( m \times m^2/4 \) simple matrix avoiding \( I_3 \),
so \( \text{forb}(m, I_3) \) is \( \Omega(m^2) \).

\( \text{forb}(m, I_3) = \binom{m}{2} + \binom{m}{1} + \binom{m}{0} \)
By considering the construction $I \times I^c$ that avoids $F_2(1, 2, 2, 1)$ and the constructions $I^c \times I^c$ or $I^c \times T$ or $T \times T$ that avoids $I_3$, we note $x(\{I_3, F_2(1, 2, 2, 1)\}) = 2$ so that we have only linear obvious constructions ($I_m^c$ or $T_m$) that avoid both $F_2(1, 2, 2, 1)$ and $I_3$. We are led to the following:

**Theorem** $\text{forb}(m, \{I_3, F_2(1, 2, 2, 1)\})$ is $\Theta(m)$. 
By considering the construction $I \times I^c$ that avoids $F_2(1, 2, 2, 1)$ and the constructions $I^c \times I^c$ or $I^c \times T$ or $T \times T$ that avoids $I_3$, we note $x(\{I_3, F_2(1, 2, 2, 1)\}) = 2$ so that we have only linear obvious constructions ($I_m^c$ or $T_m$) that avoid both $F_2(1, 2, 2, 1)$ and $I_3$. We are led to the following:

**Theorem** forb$(m, \{I_3, F_2(1, 2, 2, 1)\})$ is $\Theta(m)$.

We can extend the argument quite far:

**Theorem** forb$(m, \{t \cdot I_k, F_2(1, t, t, 1)\})$ is $\Theta(m)$. 

Forbidden Families of Configurations
Another example:

\[ \text{forb}(m, \{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 11 \cdots 1 & 00 \cdots 0 & 1 \\ 0 & 00 \cdots 0 & 11 \cdots 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 00 \cdots 0 & 11 \cdots 1 & 1 \end{bmatrix} ) \} \text{ is } O(m). \]
Another example:

\[
\begin{bmatrix}
1 & 0 \\
1 & 0 \\
0 & 1 \\
0 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1
\end{bmatrix}
\]

\(\text{forb}(m, \{\)} \} \) is \(O(m)\).

We studied the 9 ‘minimal’ configurations that have quadratic bounds and were able to verify the predictions of the conjecture for all subsets of these 9.
An unusual Bound

**Theorem** (A,Koch,Raggi,Sali 12) $\text{forb}(m, \{ T_2 \times T_2, l_2 \times l_2 \})$ is $\Theta(m^{3/2})$.

$$T_2 \times T_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad l_2 \times l_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} (= C_4)$$

We showed initially that $\text{forb}(m, \{ T_2 \times T_2, T_2 \times l_2, l_2 \times l_2 \})$ is $\Theta(m^{3/2})$ but Christina Koch realized that we ought to be able to drop $T_2 \times l_2$ and we were able to redo the proof (which simplified slightly!).
Miguel Raggi, Attila Sali
Let $A$ be an $m \times \text{forb}(m, F)$ simple matrix with no configuration in $F = \{T_2 \times T_2, I_2 \times I_2\}$. We can select a row $r$ and reorder rows and columns to obtain

$$A = \text{row } r \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & C_r & C_r & D_r \end{bmatrix}. $$
Let $A$ be an $m \times \text{forb}(m,\mathcal{F})$ simple matrix with no configuration in $\mathcal{F} = \{T_2 \times T_2, I_2 \times I_2\}$. We can select a row $r$ and reorder rows and columns to obtain

$$A = \begin{array}{cccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & C_r & C_r & D_r \end{array}.$$}

To show $\|A\|$ is $O(m^{3/2})$ it would suffice to show $\|C_r\|$ is $O(m^{1/2})$ for some choice of $r$. Our proof shows that assuming $\|C_r\| > 20m^{1/2}$ for all choices $r$ results in a contradiction. In particular, associated with $C_r$ is a set of rows $S(r)$ with $S(r) \geq 5m^{1/2}$. We let $S(r) = \{r_1, r_2, r_3, \ldots\}$. After some work we show that $|S(r_i) \cap S(r_j)| \leq 5$. Then we have

$$|S(r_1) \cup S(r_2) \cup S(r_3) \cup \cdots| = |S(r_1)| + |S(r_2)\backslash S(r_1)| + |S(r_3)\backslash (S(r_1) \cup S(r_2))| + \cdots = 5m^{1/2} + (5m^{1/2} - 5) + (5m^{1/2} - 10) + \cdots > m!!!$$
Thanks to all the organizers of CanaDAM 2013!
Great to visit Newfoundland.
I very much enjoyed the Fish and Brew(i)s.