COSTAS ARRAYS II. STRUCTURAL PROPERTIES

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ABSTRACT. We prove restrictions on when a Costas array can contain a large corner region whose entries are all 0. In particular, we prove a 2010 conjecture due to Russo, Erickson and Beard. We constrain the vectors joining pairs of 1s in a Costas array by establishing a series of results on its number of “mirror pairs,” namely pairs of these vectors having the same length but opposite slopes.

1. INTRODUCTION

This paper deals with structural properties of Costas arrays. We use definitions and notation from the companion paper [9], and results from that paper on toroidal vectors in augmented Costas arrays.

Section 2 describes a key auxiliary 1985 result, due to Freedman and Levanon [8], that two Costas arrays of order at least 4 always have a vector (joining pairs of 1s) in common. We give a complete proof of this result, which illustrates several ideas used elsewhere in the paper.

Section 3 studies Costas arrays that contain a large corner region whose entries are all 0. In the case of even order, when the corner region is an entire quadrant of the array, we use the Freedman-Levanon result to prove a 2010 conjecture due to Russo, Erickson and Beard [11].

Section 4 introduces the concept of “mirror pairs” in Costas arrays, namely pairs of vectors having the same length but opposite slopes. We again use the Freedman-Levanon result, but rather than comparing vectors contained in two different Costas arrays we constrain the vectors contained in a single Costas array. If sufficiently strong results about mirror pairs can be found, the computational burden for determining whether or not a Costas array of order 32 exists (estimated as 45,000 processor years in 2011 [5]) could be significantly reduced.

2. THE FREEDMAN-LEVANON RESULT

In this section we prove the Freedman-Levanon result on common vectors in Costas arrays, as Theorem 2.5. This result was originally stated and
proved in terms of the cross-correlation function of the Costas arrays [8]; we instead use a formulation and proof due to Drakakis, Gow and Rickard [6]. Their method studies the vectors of a Costas array by means of the difference triangle of its corresponding permutation. As in Section 1 of [9], we consider only the rightwards-pointing vector joining each pair of 1s in a Costas array, and take the first component of the vector to be horizontal.

**Definition 2.1.** The difference triangle $T(\alpha)$ of $\alpha \in S_n$ is $(t_{i,j}(\alpha))$, where $t_{i,j}(\alpha) = \alpha(i+j) - \alpha(j)$ for $1 \leq i < n$, $1 \leq j \leq n-i$.

We use the standard labelling convention for arrays (first index downwards, second index rightwards) to number the rows and columns of the difference triangle. For example, Figure 1 shows the difference triangle of the permutation $\alpha = [3, 1, 6, 2, 5, 4]$, in which row 2 is the sequence $(3, 1, -1, 2)$ and column 4 is the sequence $(3, 2)$. The antidiagonals are the sequences $(-2), (3, 5), \ldots, (1, 3, -2, 2, -1)$.

![Figure 1. Difference triangle of the permutation [3, 1, 6, 2, 5, 4]](image)

In preparation for the proof of Theorem 2.5, we give some basic properties of the difference triangle of a permutation. The combination of Lemma 2.2(i) (which is elementary) with Lemma 2.2(ii), due to Costas [2], shows that the difference triangle is a setting in which the defining property of a Costas array appears mathematically natural.

**Lemma 2.2.** Let $\alpha$ be a permutation.

(i) No column of $T(\alpha)$ and no antidiagonal of $T(\alpha)$ contains a repeated value.

(ii) No row of $T(\alpha)$ contains a repeated value if and only if the permutation array corresponding to $\alpha$ is a Costas array.

**Proof.** Part (i) holds because $\alpha$ is a permutation. Part (ii) holds because the permutation array corresponding to $\alpha$ contains the vector $(w, h)$ starting from position $(\alpha(j), j)$ if and only if $t_{w,j}(\alpha) = h$. □

**Lemma 2.3** ([3]). The difference triangle of $\alpha \in S_n$ contains exactly $n-k$ entries from $\{-k, k\}$, for each $k$ satisfying $1 \leq k < n$.

**Proof.** From the $n$ entries of $\alpha$ we can form exactly $n-k$ pairs whose values differ in magnitude by $k$ (namely the pairs $\{\ell, k + \ell\}$ for $1 \leq \ell \leq n-k$). □
The following lemma specifies relations between certain elements of the difference triangle. It follows directly from the definition of the difference triangle and, for part (i), from the permutation property of $\alpha$. A more general version of the lemma is given in [13, Section 4.2].

**Lemma 2.4.** Let $\alpha \in S_n$. Then

(i) for $n \geq 3$ and $1 \leq c \leq n - 2$,

$$t_{n-1,1}(\alpha) = t_{n-1-c,1}(\alpha) + t_{n-1-c,1+c}(\alpha)$$

if and only if $n = 2c + 1$

(ii) for $n \geq 4$,

$$t_{n-3,2}(\alpha) = t_{n-2,1}(\alpha) + t_{n-2,2}(\alpha) - t_{n-1,1}(\alpha).$$

**Theorem 2.5** (Freedman-Levanon [8]). Every pair of Costas arrays of order $n \geq 4$ has a vector in common.

**Proof.** [6] Suppose, for a contradiction, that $\alpha, \beta$ are permutations corresponding to Costas arrays of order $n \geq 4$ having no vector in common. For $1 \leq w < n$, let $w(\alpha)$ and $w(\beta)$ be the set of elements contained in row $w$ of $T(\alpha)$ and $T(\beta)$, respectively, and let $w(\alpha, \beta)$ be the multiset union of $w(\alpha)$ and $w(\beta)$. By assumption and by Lemma 2.2(ii), for each $w$ the multiset $w(\alpha, \beta)$ has no repeated elements and so is actually a set.

For $1 \leq k < n$, we now prove by induction on $k$ that

$$-k, k \in w(\alpha, \beta) \quad \text{for} \quad w = 1, 2, \ldots, n - k.$$  \hspace{1cm} (2.1)

For the base case $k = 1$, we know from Lemma 2.3 that there are a total of $2(n - 1)$ entries from $\{-1, 1\}$ distributed over the $n - 1$ sets $\{w(\alpha, \beta) : 1 \leq w \leq n - 1\}$. Since each of these sets contains no repeated elements, we conclude that $-1, 1 \in w(\alpha, \beta)$ for $w = 1, 2, \ldots, n - 1$. This establishes the base case. Assume now that the cases up to $k - 1$ hold. By Lemma 2.3, there are a total of $2(n - k)$ entries from $\{-k, k\}$ distributed over the $n - 1$ sets $\{w(\alpha, \beta) : 1 \leq w \leq n - 1\}$. But the inductive hypothesis implies that the sets $\{w(\alpha, \beta) : n - k \leq w \leq n - 1\}$ contain no elements from $\{-k, k\}$, so the $2(n - k)$ entries from $\{-k, k\}$ are in fact distributed over the $n - k$ sets $\{w(\alpha, \beta) : 1 \leq w \leq n - k\}$. Since each of these sets contain no repeated elements, we conclude that $-k, k \notin w(\alpha, \beta)$ for $w = 1, 2, \ldots, n - k$. This completes the induction.

It follows from (2.1) that $t_{n-k}(\alpha, \beta) = \{-k, \ldots, -1, 1, \ldots, n - k\}$ for $1 \leq k < n$ and, in particular, that

$$t_{n-1}(\alpha, \beta) = \{-1, 1\},$$

$$t_{n-2}(\alpha, \beta) = \{-2, -1, 1, 2\},$$

$$t_{n-3}(\alpha, \beta) = \{-3, -2, -1, 1, 2, 3\}.$$  \hspace{1cm} (2.2)

(2.3)

We may assume from (2.2) that $t_{n-1}(\alpha) = \{1\}$ and $t_{n-1}(\beta) = \{-1\}$. Lemma 2.2(ii) then implies that $1 \notin t_{n-2}(\alpha)$ and $-1 \notin t_{n-2}(\beta)$. Now $t_{n-2}(\alpha) \neq \{-2, 2\}$ since otherwise (2.3) would imply the contradiction $-1 \in t_{n-2}(\beta)$, and $t_{n-2}(\alpha) \neq \{-1, 2\}$ otherwise Lemma 2.4(i) with $c = 1$ would imply the contradiction
\( n = 3 \). Therefore \( t_{n-2}(\alpha) = \{-2, -1\} \), by (2.3). Finally, Lemma 2.4(ii) shows that \( t_{n-3,2}(\alpha) = -2 - 1 - 1 = -4 \), contradicting (2.4).

3. LARGE ALL-ZERO CORNER REGIONS

**Definition 3.1.** An all-zero corner region of a permutation array \( A \) is a square subarray, whose entries are all 0, containing one of the four corner elements of \( A \).

The size of an all-zero corner region of a permutation array of order \( n \) is at most \( \left\lfloor \frac{n}{2} \right\rfloor \times \left\lfloor \frac{n}{2} \right\rfloor \), by the permutation property. In this section, we study Costas arrays that attain this upper bound. We show for even \( n \) that this occurs only for small orders (Theorem 3.3), and for odd \( n \) that this strongly constrains the distribution of 1s in the array (Theorem 3.7).

We begin with even \( n \), when a corner region of size \( \frac{n}{2} \times \frac{n}{2} \) is simply a quadrant. Figure 2 shows examples of Costas arrays of order 2, 4 and 6 containing two diagonally opposite all-zero quadrants. It was conjectured in 2010 that there are no larger such examples.

**Conjecture 3.2** (Russo, Erickson and Beard [11]). No Costas array of even order greater than 6 contains two all-zero quadrants.

![Figure 2. Costas arrays with two all-zero quadrants](image)

We now use Theorem 2.5 to prove a result stronger than the statement of Conjecture 3.2.

**Theorem 3.3.** No Costas array of even order greater than 6 contains an all-zero quadrant.

**Proof.** Suppose, for a contradiction, that \( A \) is a Costas array of order \( 2m > 6 \) containing an all-zero quadrant. Since two diagonally opposite quadrants of a permutation array of even order must contain equally many 1s, the array \( A \) has two all-zero quadrants in diagonally opposite positions. Each of the other two quadrants then forms a Costas array of order \( m > 3 \). By Theorem 2.5, these two Costas arrays contain a common vector. This contradicts the Costas property for \( A \).

**Corollary 3.4** (Conjecture 3.2 holds). No Costas array of even order greater than 6 contains two all-zero quadrants.
We now consider odd \( n \), when an all-zero corner region attaining the upper bound has size \( \frac{n-1}{2} \times \frac{n-1}{2} \). We use the following counterpart to Theorem 2.5, which is proved by similar methods.

**Theorem 3.5** (Drakakis, Gow and Rickard [6, Theorem 4.3]). Every pair of Costas arrays of orders \( m \) and \( m + 1 \), where \( m > 2 \), has a vector in common.

**Proposition 3.6.** A Costas array of odd order \( n > 5 \) has at most one all-zero corner region of size \( \frac{n-1}{2} \times \frac{n-1}{2} \).

**Proof.** Suppose, for a contradiction, that \( A \) is a Costas array of odd order \( n > 5 \) having two all-zero corner regions \( R_1 \) and \( R_3 \), as depicted in Figure 3. We may take the number of 1s in both regions \( Y \) and \( Z \) to be 0; otherwise, we may assume by reflection in the diagonal or antidiagonal of \( A \) that these numbers are 1 and 0, respectively, and then a count of 1s in the last \( \frac{n+1}{2} \) rows of \( A \) and the first \( \frac{n+1}{2} \) columns of \( A \) contradicts the permutation property. Therefore \( A \) contains a Costas array of order \( \frac{n-1}{2} > 2 \) in region \( R_2 \), and a Costas array of order \( \frac{n+1}{2} \). By Theorem 3.5, these two Costas arrays contain a common vector, which contradicts the Costas property for \( A \). \( \square \)

\[ \begin{array}{ccc}
\frac{n-1}{2} & 1 & \frac{n-1}{2} \\
\hline
\frac{n-1}{2} & R_1 & Y \\
1 & X & Z \\
\frac{n-1}{2} & R_4 & R_3 \\
\end{array} \]

**Figure 3.** Array regions for Proposition 3.6 and Theorem 3.7

**Theorem 3.7.** Suppose that \( A \) is a Costas array of odd order \( n > 5 \) containing an all-zero corner region \( R_1 \) of size \( \frac{n-1}{2} \times \frac{n-1}{2} \), as depicted in Figure 3. Then the number of 1s in each of the regions \( R_2 \) and \( R_4 \) is \( \frac{n-3}{2} \), and the number of 1s in each of the regions \( R_3 \), \( X \) and \( Y \) is 1.

**Proof.** Region \( R_3 \) contains at least one ‘1’ entry, by Proposition 3.6. Region \( Y \) therefore contains a ‘1’ entry; otherwise, a count of 1s in the first \( \frac{n-1}{2} \) rows of \( A \) and the last \( \frac{n-1}{2} \) columns of \( A \) contradicts the permutation property. Then consider the first \( \frac{n-1}{2} \) rows of \( A \) to show that region \( R_2 \) contains exactly \( \frac{n-3}{2} \) 1s, and consider the last \( \frac{n-1}{2} \) columns of \( A \) to show that region \( R_3 \) contains exactly one 1. The result follows by symmetry. \( \square \)
The database [10] of Costas arrays up to order 29 shows that, up to equivalence, the number of Costas arrays of odd order \( n \) containing an all-zero corner region of size \( \frac{n-1}{2} \times \frac{n-1}{2} \) is one for \( n = 3 \), five for \( n = 5 \), eight for \( n = 7 \), two for \( n = 9 \), and zero for \( 11 \leq n \leq 29 \). Figure 4(a) shows a Costas array of order 5 containing two such corner regions (note that order 5 is excluded in Proposition 3.6), and Figure 4(b) shows a Costas array of order 9 with one such corner region.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{costas_array}
\caption{Costas arrays of odd order \( n \) with an all-zero corner region of size \( \frac{n-1}{2} \times \frac{n-1}{2} \)}
\end{figure}

4. Mirror Pairs

Transformation of a Costas array under the action of the dihedral group \( D_4 \) gives an equivalent Costas array. By applying Theorem 2.5 to a Costas array \( A \) of order at least 4, and its image under reflection in a vertical axis, we can conclude that \( A \) must contain a pair of related vectors. (We can guarantee additional pairs of related vectors by applying Theorem 2.5 to \( A \) and its image under 90° or 270° rotation, but not necessarily to \( A \) and its image under diagonal or antidiagonal reflection.) This motivates the following definition and proposition.

**Definition 4.1.** A Costas array contains a mirror pair of width \( w > 0 \) and height \( h > 0 \) (abbreviated as a \((w, h)\)-mirror pair) if it contains vectors \((w, h)\) and \((w, -h)\).

**Proposition 4.2.** Every Costas array of order \( n \geq 4 \) contains a mirror pair.

**Proof.** Let \( A \) be a Costas array of order \( n \geq 4 \). By Theorem 2.5, \( A \) and its image under reflection in a vertical axis have a common vector, say \((w, h)\). Then \( A \) contains the vector \((w, -h)\). \( \square \)
For example, the Costas array of order 7 shown in Figure 5 contains a 
(3,1)-mirror pair. The condition \( n \geq 4 \) in Proposition 4.2 is necessary: the 
endpoints of the vectors \((w, h)\) and \((w, -h)\) must involve distinct ‘1’ entries, 
otherwise the permutation property of the Costas array would not hold.

![Figure 5. A (3,1)-mirror pair in a Costas array of order 7](image)

The existence of mirror pairs in Costas arrays is a structural property 
that does not appear to have been observed previously. In this section, we 
constrain the vectors of a Costas array by analysing the number of its mirror 
pairs and their parameters \( w \) and \( h \).

We begin with an upper bound on the number of mirror pairs of width \( w \).

**Proposition 4.3.** Let \( A \) be a Costas array of order \( n \geq 4 \) containing \( m_w \) 
mirror pairs of width \( w \). Then

(i) \( m_1 \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \).

(ii) \( m_w \leq \left\lfloor \frac{n-w}{2} \right\rfloor \) for each \( w \) satisfying \( 2 \leq w \leq n - 1 \).

(iii) The total number of mirror pairs in \( A \) is at most \( \frac{n(n-2)}{4} \) for \( n \) even, 
and at most \( \frac{(n+1)(n-3)}{4} \) for \( n \) odd.

**Proof.** For parts (i) and (ii), let \( 1 \leq w \leq n - 1 \). Since \( A \) is a permutation 
array, it contains exactly \( n - w \) vectors with width (first component) \( w \) and 
therefore at most \( \left\lfloor \frac{n-w}{2} \right\rfloor \) mirror pairs of width \( w \). It is now sufficient to show 
that \( A \) cannot contain \( \frac{n-1}{2} \) mirror pairs of width 1 when \( n \) is odd. Suppose 
otherwise, so that the \( n - 1 \) vectors of \( A \) with width 1 can be arranged into 
mirror pairs of height \( h_1, h_2, \ldots, h_{\frac{n+1}{2}} \), and let the ‘1’ entry in column 1 
of \( A \) occur in row \( i \). Then the ‘1’ entry in column \( n \) of \( A \) occurs in row 
\( i + \sum_k h_k + \sum_k (-h_k) = i \), contradicting the permutation property.

Part (iii) is given by summing the bounds of parts (i) and (ii) over \( w \). \( \square \)

Numerical data obtained from analysis of the database [10] of Costas 
arrays, presented in Figure 6, suggest that the actual number of mirror
pairs in a Costas array of order \( n \) broadly increases with \( n \), and that its mean grows faster than linearly with \( n \). In Section 4.1 we shall strengthen Proposition 4.3 for G-symmetric Costas arrays by using their additional structure to establish lower and upper bounds on the number of mirror pairs of various widths. In Section 4.2 we shall fix precisely the number of mirror pairs of every width in Welch Costas arrays. In Section 4.3 we shall prove mirror pair existence results for Golomb Costas arrays, using results from [9].

The large number of vectors having small width suggests that small values of \( w \) might be more likely to admit mirror pairs of width \( w \). This leads us to pay particular attention to mirror pairs of small width and, by similar reasoning, those of small height. Analysis of the database [10] of Costas arrays shows that there is at least one width 1 mirror pair and at least one height 1 mirror pair in each Costas array with \( 4 \leq n \leq 8 \), with the exception of the Costas arrays corresponding to the permutations \([3, 1, 2, 4], [2, 3, 5, 1, 4], [2, 6, 4, 5, 1, 3], [5, 3, 2, 6, 1, 4], [2, 5, 1, 6, 4, 3] [3, 1, 6, 3, 5, 4], [2, 6, 3, 8, 1, 7, 5, 4] and [2, 8, 1, 6, 5, 3, 7, 4] and their equivalence classes. (Of these, only that corresponding to \([5, 3, 2, 6, 1, 4]\) lacks both a width 1 mirror pair and a height 1 mirror pair.) Moreover, there is at least one width 1 mirror pair and at least one height 1 mirror pair in every Costas array with \( 9 \leq n \leq 29 \). These observations prompt the following question.

**Question 4.4.** Does every Costas array of order \( n \geq 9 \) contain a mirror pair of width 1 and a mirror pair of height 1?
We can simplify Question 4.4 by noting the action of $D_4$ on the mirror pairs of a Costas array.

**Remark 4.5.** Suppose that $A$ contains a $(w,h)$-mirror pair. Then so does its image under horizontal reflection, vertical reflection and rotation by $180^\circ$. Its image under diagonal reflection, antidiagonal reflection and rotation by $90^\circ$ and $270^\circ$ contains an $(h,w)$-mirror pair.

We see that in order to answer Question 4.4 with yes, it would be sufficient to show that there is at least one width 1 mirror pair in each Costas array of order $n \geq 9$ (not just in the equivalence class of each Costas array). Indeed, by Remark 4.5, there would then also be at least one height 1 mirror pair in each such Costas array. Numerical data presented in Figure 7 suggest that the number of width 1 mirror pairs in a Costas array of order $n$ grows with $n$, providing evidence that the answer to Question 4.4 is yes. Figure 7 also shows that the upper bound on the number $m_1$ of width 1 mirror pairs, given in Proposition 4.3(i), is attained for all $n$ in the range $4 \leq n \leq 29$ except 24, 25, and 26. We shall see in Theorem 4.9(i) that all $G$-symmetric Costas arrays of order $n$ attain this upper bound.

![Figure 7. Number $m_1$ of width 1 mirror pairs in Costas arrays up to order 29](image)

Analysis of the Costas array database [10] also shows that for $14 \leq n \leq 29$ every Costas array of order $n$ has a mirror pair of width 2 and therefore, by Remark 4.5, a mirror pair of height 2. This prompts the following question.

**Question 4.6.** Does every Costas array of order $n \geq 14$ have a mirror pair of width 2 and a mirror pair of height 2?
Questions 4.4 and 4.6 are more easily studied for Costas arrays that are algebraically constrained (G-symmetric Costas arrays) or constructed (Welch Costas arrays and Golomb Costas arrays). We will examine the mirror pairs of small width and height in these classes of Costas arrays in Sections 4.1, 4.2 and 4.3. Before doing so, we will provide a partial answer to Questions 4.4 and 4.6 for all Costas arrays, without imposing any such algebraic restrictions, in Theorem 4.7.

Some of our proofs make extensive use of the difference triangle. By Definition 2.1, a Costas array with corresponding permutation $\alpha$ contains a $(w, h)$-mirror pair if and only if both $-h$ and $h$ appear in row $w$ of $T(\alpha)$.

**Theorem 4.7.** Every Costas array of order $n \geq 6$ contains a mirror pair of width 1 or 2 and a mirror pair of height 1 or 2.

**Proof.** By Remark 4.5, it is sufficient to show that every Costas array of order $n \geq 6$ contains a mirror pair of height 1 or 2. Suppose, for a contradiction, that $A$ is a Costas array of order $n \geq 6$, with corresponding permutation $\alpha$, containing neither a mirror pair of height 1 nor a mirror pair of height 2. Then no row of $T(\alpha)$ contains more than one entry from $\{-1, 1\}$ and no row of $T(\alpha)$ contains more than one entry from $\{-2, 2\}$. Since $T(\alpha)$ contains exactly $n-1$ entries from $\{-1, 1\}$ by Lemma 2.3, it therefore contains exactly one entry from $\{-1, 1\}$ in each of its $n-1$ rows. This accounts for the single entry of row $n-1$ of $T(\alpha)$; and since $T(\alpha)$ contains exactly $n-2$ entries from $\{-2, 2\}$ by Lemma 2.3, it must then contain exactly one entry from $\{-2, 2\}$ in each of its first $n-2$ rows.

Write row $i$ of $T(\alpha)$ as $r_i(\alpha)$. By Remark 4.5, both rotation of $A$ through 180° and reflection of $A$ in a horizontal axis leave the mirror pairs of $A$ unchanged, but the first transformation reflects the rows of $T(\alpha)$ and the second negates the entries of $T(\alpha)$. We may therefore assume that $r_{n-1}(\alpha) = (1)$ and $r_{n-2}(\alpha) = (x, y)$, where $x \in \{-1, 1\}$ and $y \in \{-2, 2\}$. Lemma 2.2(i) then gives $x = -1$, and Lemma 2.4(i) with $c = 1$ gives $y = -2$. Then by Lemma 2.4(ii), $r_{n-3}(\alpha) = (u, -4, v)$, where $\{|u|, |v|\} = \{1, 2\}$. By Lemma 2.2(i) we have $u \notin \{-1, 1\}$ and $v \neq 1$, so $(u, v) = (2, -1)$ or $(-2, -1)$. But by Lemma 2.4(i) with $c = 2$, we have $u + v \neq 1$ since $n \geq 6$. This forces $(u, v) = (-2, -1)$, so the last three rows of $T(\alpha)$ are as shown below.

$$
\begin{array}{ccc}
-2 & -4 & -1 \\
-1 & -2 \\
1 \\
\end{array}
$$

From these entries of $T(\alpha)$ and the assumption $n \geq 6$ we obtain $\alpha = [m, m + 3, m + 2, \ldots, m - 2, m - 1, m + 1]$ for some $m$. Then the first row of $T(\alpha)$ contains both $(m + 2) - (m + 3) = -1$ and $(m - 1) - (m - 2) = 1$, which is a contradiction. \qed
By examining all Costas arrays of order 4 and 5, we can extend Theorem 4.7 to \( n \geq 4 \), with the exception of the Costas array corresponding to the permutation \([2, 3, 5, 1, 4]\) and its equivalence class.

4.1. **Mirror pairs in G-symmetric Costas arrays.** In this section, we consider mirror pairs in G-symmetric Costas arrays of order \( n \) (as defined in [9, Definition 2.3]). We show the answer to Question 4.4 for this class of Costas arrays is *yes* for even \( n \) (Theorem 4.9(i) and Theorem 4.12). We also provide a partial answer to Question 4.4 for odd \( n \) by showing the existence of a width 1, but not necessarily a height 1, mirror pair for arrays in this class (Theorem 4.9(ii)). We likewise provide a partial answer to Question 4.6 for all \( n \) for arrays in this class (Theorem 4.9(ii)). Furthermore, we constrain the total number of mirror pairs in a G-symmetric Costas array more strongly than in Proposition 4.3(iii) (Theorem 4.9(v)) by proving new lower and upper bounds for various widths.

We call ‘1’ entries of a G-symmetric array that are separated by \( \left\lfloor \frac{n+1}{2} \right\rfloor \) columns **G-symmetric images** of each other. We require the following lemma in the proof of Theorem 4.9.

**Lemma 4.8.** Let \( G \) be a G-symmetric Costas array of order \( n \geq 4 \), and let \( L \) and \( R \) denote the leftmost \( \left\lfloor \frac{n}{2} \right\rfloor \) and rightmost \( \left\lfloor \frac{n}{2} \right\rfloor \) columns of \( G \), respectively.

(i) Every vector joining ‘1’ entries in \( L \) forms a mirror pair with the vector in \( R \) that joins the G-symmetric images of its endpoints.

(ii) If one vector of a mirror pair in \( G \) joins a ‘1’ entry in \( L \) to a ‘1’ entry in \( R \) then so does the other. The G-symmetric images of the endpoints of two such vectors that form a \((w, h)\)-mirror pair are the endpoints of two such vectors that form a \((2 \left\lfloor \frac{n+1}{2} \right\rfloor - w, h)\)-mirror pair.

**Proof.** For part (i), let \( 1 \leq w \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \) and \( 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor - w \). By G-symmetry, the vector in \( L \) joining the ‘1’ entries in columns \( j \) and \( j + w \) forms a mirror pair of width \( w \) with the vector in \( R \) joining the ‘1’ entries in columns \( j + \left\lfloor \frac{n+1}{2} \right\rfloor \) and \( j + w + \left\lfloor \frac{n+1}{2} \right\rfloor \). Part (ii) follows directly from G-symmetry. \( \Box \)

**Theorem 4.9.** Let \( G \) be a G-symmetric Costas array of order \( n \geq 4 \) containing \( m_w \) mirror pairs of width \( w \). Then

(i) \( m_1 = \left\lfloor \frac{n}{2} \right\rfloor - 1 \).

(ii) \( m_w \geq \left\lfloor \frac{n}{2} \right\rfloor - w \) for each \( w \) satisfying \( 2 \leq w \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \).

(iii) \( m_w \leq \left\lfloor \frac{n-w}{2} \right\rfloor \) for each \( w \) satisfying \( 2 \leq w \leq n - 1 \).

(iv) \( m_{\left\lfloor \frac{n+1}{2} \right\rfloor} = 0 \).

(v) For even \( n \), the total number of mirror pairs in \( G \) lies in the interval 

\[
\left\lfloor \frac{n(n-2)}{8} \right\rfloor, \left\lfloor \frac{n(n-2)}{4} - \left\lfloor \frac{n}{4} \right\rfloor \right]\]
and has the same parity as \( \frac{n(n-2)}{8} \). For odd \( n \), the total number of mirror pairs in \( G \) lies in the interval
\[
\left[ \frac{(n-1)(n-3)}{8}, \frac{(n+1)(n-3)}{4} - \left\lfloor \frac{n-1}{4} \right\rfloor \right],
\]

Proof. For parts (i) and (ii), let \( 1 \leq w \leq \left\lfloor \frac{n^2}{2} \right\rfloor - 1 \). Lemma 4.8(i) accounts for \( \left\lfloor \frac{n^2}{2} \right\rfloor - w \) mirror pairs of width \( w \) to give part (ii), and combination with Proposition 4.3(i) gives part (i).

Part (iii) restates Proposition 4.3(ii).

For part (iv), since the endpoints of a vector of width \( \left\lfloor \frac{n+1}{2} \right\rfloor \) in \( G \) are \( G \)-symmetric images of each other, their vertical separation varies with the smaller of the row numbers in which they occur. Therefore no two distinct vectors of width \( \left\lfloor \frac{n+1}{2} \right\rfloor \) in \( G \) have endpoints with the same vertical separation, and so \( G \) does not contain a mirror pair of width \( \left\lfloor \frac{n+1}{2} \right\rfloor \).

For part (v), the intervals for the total number of mirror pairs in \( G \) are given by combining all previous parts. For even \( n \), the lower limit of the interval is a count via Lemma 4.8(i) of the mirror pairs involving a vector joining ‘1’ entries in the left half of \( G \) and a vector joining ‘1’ entries in the right half of \( G \). Since this count exhausts every such vector, all other mirror pairs must involve vectors joining a ‘1’ entry in the left half of \( G \) to a ‘1’ entry in the right half of \( G \). Then by Lemma 4.8(ii), these additional mirror pairs can be associated using \( G \)-symmetry into pairs, and by part (iv) no mirror pair is associated with itself. This gives the parity condition.

The upper bound given in Theorem 4.9(v) on the total number of mirror pairs in a \( G \)-symmetric Costas array of order \( n \) is strictly larger than the lower bound for all \( n > 4 \). This upper bound is attained by a \( G \)-symmetric Costas array of order \( n \) in the case \( n = 5 \) (corresponding permutation \([2, 5, 3, 4, 1]\), total two mirror pairs), the case \( n = 6 \) (corresponding permutation \([1, 5, 3, 6, 2, 4]\), total five mirror pairs), the case \( n = 8 \) (corresponding permutations \([1, 8, 6, 3, 7, 2, 4, 5]\), \([1, 6, 7, 4, 8, 3, 2, 5]\), \([7, 1, 2, 8, 4, 6, 5, 3]\), total ten mirror pairs each), but in no other case in the range \( 8 < n \leq 28 \) (by reference to the Costas array database [1] for \( n < 28 \), and [5], [12] for \( n = 28 \)).

We noted previously that if every Costas array of order \( n \) contains a width 1 mirror pair then, by Remark 4.5, it also contains a height 1 mirror pair. However, we cannot conclude from Theorem 4.9(i) that every \( G \)-symmetric Costas array of order \( n \geq 4 \) contains even a single mirror pair of height 1, because \( G \)-symmetry is not preserved under the transpose operation. Nonetheless, we shall prove in Theorem 4.12 that (apart from some small exceptions) every \( G \)-symmetric Costas array of even order indeed contains a height 1 mirror pair. We firstly establish two lemmas, in preparation for a proof by contradiction.

**Lemma 4.10.** Suppose that \( G \) is a \( G \)-symmetric Costas array of even order \( n \), corresponding to the permutation \( \gamma \) and containing no mirror pair of height 1. Then the first \( \frac{n}{2} \) elements of \( \gamma \) all have the same parity.
Proof. No vector of $G$ of height 1 is completely contained in the left half or the right half of $G$, otherwise by Lemma 4.8(i) $G$ would contain a mirror pair of height 1. Therefore no two ‘1’ entries in the left half of $G$ occur in consecutive rows, and no two ‘1’ entries in the right half of $G$ occur in consecutive rows. It follows that the ‘1’ entry in row $i$ of $G$ occurs in the opposite half from the ‘1’ entry in row $i+1$. □

We next constrain the difference triangle of a permutation $\gamma$ satisfying the parity constraint in Lemma 4.10.

Lemma 4.11. Let $G$ be a $G$-symmetric Costas array of even order $n$, corresponding to the permutation $\gamma$, and suppose that the first $\frac{n}{2}$ entries of $\gamma$ all have the same parity. Let $T_1$ and $T_2$ be the triangular regions of $T(\gamma)$ indicated below, each involving $\frac{n}{2}$ rows.

Then

(i) all entries in $T_1$ are even, and $T_2 = -T_1$
(ii) for $1 \leq w \leq \frac{n}{2} - 1$, row $w$ of $T_1$ contains exactly one element from each of $\{-2, 2\}, \{-4, 4\}, \ldots, \{-n+2w, n-2w\}$.

Proof. For (i), let $1 \leq w \leq \frac{n}{2} - 1$ and $1 \leq j \leq \frac{n}{2} - w$. The $(w,j)$ entry of $T_1$ is $\gamma(w+j) - \gamma(j)$, which is even because $\gamma(w+j)$ and $\gamma(j)$ have the same parity. The $(w,j)$ entry of $T_2$ is $\gamma(w+j + \frac{n}{2}) - \gamma(j + \frac{n}{2}) = (n+1 - \gamma(w+j)) - (n+1 - \gamma(j))$ by $G$-symmetry, and so $T_2 = -T_1$.

For (ii), let $1 \leq w \leq \frac{n}{2} - 1$. For each $k$ satisfying $1 \leq k \leq \frac{n}{2} - 1$, by Lemma 2.3 the difference triangle $T(\gamma)$ contains exactly $n-2k$ elements from $\{-2k, 2k\}$. There are a total of $\frac{n}{2}(\frac{n}{2} - 1)$ of these even elements in $T(\gamma)$, and by (i) they are all contained in $T_1 \cup T_2$. We deduce from $T_2 = -T_1$ that, for each $k$ satisfying $1 \leq k \leq \frac{n}{2} - 1$, the triangle $T_1$ contains exactly $\frac{n}{2} - k$ elements from $\{-2k, 2k\}$ distributed over its $\frac{n}{2} - 1$ rows. By Lemma 2.2(ii) and $T_2 = -T_1$, no two such elements occur in the same row of $T_1$. The result now follows by a simple induction. □

We can now classify the $G$-symmetric Costas arrays of even order $n \geq 4$ that do not contain a mirror pair of height 1.
Theorem 4.12. The only G-symmetric Costas arrays of even order \( n \geq 4 \) that do not contain a mirror pair of height 1 are those corresponding to the permutations \([3, 1, 2, 4]\) and \([2, 6, 4, 5, 1, 3]\) and their images under horizontal reflection, vertical reflection and \(180^\circ\) rotation.

Proof. Suppose that \( G \) is a G-symmetric Costas array of even order \( n \geq 4 \), corresponding to the permutation \( \gamma \) and containing no mirror pair of height 1. By Lemma 4.10, the conclusions of Lemma 4.11 hold.

By Lemma 2.3, \( T(\gamma) \) has exactly \( n - 1 \) entries from \( \{-1, 1\} \), and by assumption no two are in the same row. Therefore

\[
\text{(4.1)} \quad \text{each row of } T(\gamma) \text{ contains exactly one entry from } \{-1, 1\}.
\]

Since horizontal reflection, vertical reflection and \(180^\circ\) rotation preserve G-symmetry, we may assume that \( t_{n-1,1}(\gamma) = 1 \) and, using Lemma 2.2(i), that \( t_{n-2,1}(\gamma) = -1 \). Write \( \gamma(1) = m \), so that \( \gamma(n-1) = m - 1 \) and \( \gamma(n) = m + 1 \). Then \( t_{1,n-1}(\gamma) = 2 \), which by Lemma 4.11(i) gives \( t_{1,\frac{n}{2}-1}(\gamma) = -2 \).

For \( n = 4 \), this last conclusion reduces to \( t_{1,1}(\gamma) = -2 \) and so \( \gamma = [m, m-2, m-1, m+1] \), which forces \( \gamma = [3, 1, 2, 4] \). For \( n > 4 \), Lemmas 4.11(ii) and 2.2(i) together give \( t_{\frac{n}{2}-1,1}(\gamma) = 2 \), and then Lemma 4.11(i) gives \( t_{\frac{n}{2}-1,\frac{n}{2}+1}(\gamma) = -2 \). For \( n = 6 \), this implies that \( \gamma = [m, m+4, m+2, m+3, m-1, m+1] \), which forces \( \gamma = [2, 6, 4, 5, 1, 3] \). It is easily verified that in both these cases \( n = 4 \) and \( n = 6 \), \( G \) is a G-symmetric Costas array but its transpose is not, giving the eight exceptional Costas arrays.

Otherwise, for \( n \geq 8 \), we seek a contradiction. Write \( x = t_{1,1}(\gamma) \) and \( y = t_{n-3,2}(\gamma) \) (see Figure 8). By Lemma 2.2(ii) we have \( x \neq -2 \), so \( \gamma(2) \neq m-2 \). Then \( y = \gamma(n-1) - \gamma(2) \neq 1 \), and therefore \( t_{n-3,3}(\gamma) = -1 \) by (4.1) and repeated use of Lemma 2.2(i). This implies that \( \gamma(3) = m+2 \), and so \( t_{2,1}(\gamma) = 2 \). This contradicts Lemma 2.2(i) because \( 2 \neq \frac{n}{2}-1 \).

\[ \square \]

4.2. Mirror pairs in Welch Costas arrays. In this section, we use results from Section 4.1 to show the answer to Question 4.4 for Welch Costas arrays is yes (Theorem 4.9(i) and Corollary 4.13). We then show that all Welch Costas arrays of order \( p-1 \geq 4 \) contain exactly \( \frac{(p-1)(p-3)}{8} \) mirror pairs (Theorem 4.14). We establish further restrictions on the mirror pairs of small width and height in a Welch Costas array \( W \) by using results from [9] on the toroidal vectors of the augmented array \( W^+ \).

Recall that every Welch Costas array is G-symmetric [7]. We firstly show that the answer to Question 4.4 for Welch Costas arrays is yes, by combination of Theorem 4.9(i) with the following corollary.

Corollary 4.13. For \( p \geq 5 \), every \( W_1(p, \phi, c) \) Welch Costas array contains at least one mirror pair of height 1 unless \((p, \phi, c)\) belongs to the set

\[ \{(5, 2, 3), (5, 2, 1), (5, 3, 2), (5, 3, 0), (7, 3, 2), (7, 3, 5), (7, 5, 5), (7, 5, 2)\} \]

Proof. Let \( W \) be a \( W_1(p, \phi, c) \) Costas array of order \( p-1 \geq 4 \). Let \( G \) and \( G' \) be the Costas arrays corresponding to the permutations \([3, 1, 2, 4]\) and
Figure 8. Difference triangle $T(\gamma)$ for the proof of Theorem 4.12

$[2, 6, 4, 5, 1, 3]$, respectively. By Theorem 4.12, $W$ contains at least one mirror pair of height 1 provided it is not the image of $G$ or $G'$ under one of following four elements of $D_4$: the identity, horizontal reflection, vertical reflection and 180° rotation. It can be verified that the parameter sets $(5, 2, 3)$, $(5, 2, 1)$, $(5, 3, 2)$ and $(5, 3, 0)$ yield the four images of $G$, respectively, and the parameter sets $(7, 3, 2)$, $(7, 3, 5)$, $(7, 5, 5)$ and $(7, 5, 2)$ yield the four images of $G'$, respectively.

All G-symmetric Costas arrays of even order $n$ contain at least $n(n-2)/8$ mirror pairs, as stated in Theorem 4.9(v). Although this minimum number can be exceeded (see Section 4.1 for examples with $n = 6$ and $n = 8$), we now show that it cannot be exceeded for the subclass of Welch Costas arrays.

**Theorem 4.14.** Every Welch Costas array of order $p-1 \geq 4$ contains exactly $\frac{(p-1)(p-3)}{8}$ mirror pairs, namely those specified in Lemma 4.8(i).

**Proof.** Let $W$ be a $W_1(p, \phi, c)$ Welch Costas array of order $p-1 \geq 4$, so that $W$ contains all $\frac{(p-1)(p-3)}{8}$ mirror pairs specified in Lemma 4.8(i). Suppose, for a contradiction, that $W$ also contains some other $(w, h)$-mirror pair. By Lemma 4.8(ii), both vectors of the mirror pair cross the vertical bisector of $W$, and by Lemma 4.8(ii) and Theorem 4.9(iv) we may take $w < \frac{p-1}{2}$.

Let the leftmost of the right endpoints of the two vectors of the $(w, h)$-mirror pair occur in column $j > \frac{p-1}{2}$. The array $W'$ obtained by cyclically shifting the columns of $W$ by $j - \frac{p-1}{2}$ places to the left then contains a $(w, h)$-mirror pair exactly one of whose vectors crosses its vertical bisector. But $W'$ is the Welch Costas array $W_1(p, \phi, c + j - \frac{p-1}{2})$, which by the same argument as given above cannot contain such a mirror pair. \qed
We shall use [9, Theorem 4.2] and Lemma 4.15 below to prove in Theorem 4.16 that every Welch Costas array has a \((w, h)\)-mirror pair with \(w + h \leq 3\). In Section 4.3, we will prove analogous results for Golomb Costas arrays. The proof of Lemma 4.15 combines results from [4, Theorem 6] that we derive here using similar methods.

**Lemma 4.15.** For \(p \geq 5\), every \(W_1(p, \phi, c)\) Welch Costas array contains a vector from \(\{(1, 1), (2, 1)\}\) and a vector from \(\{(1, -1), (2, -1)\}\).

**Proof.** Let \(W\) be a \(W_1(p, \phi, c)\) Welch Costas array with \(p \geq 5\), and let \(W^+\) be the augmented array obtained by adding a row of 0s on the bottom of \(W\). We firstly show that \(W\) contains at least one of the vectors \((1, 1)\) and \((2, 1)\). Suppose, for a contradiction, that \(W\) contains neither of these vectors. By [9, Theorem 4.2], \(W^+\) contains both of the toroidal vectors \((1, 1)\) and \((2, 1)\). Since the last row of \(W^+\) does not contain any 1s, this implies that the height 1 toroidal vectors \((1, 1)\) and \((2, 1)\) both cross the vertical boundary of \(W^+\). Therefore the toroidal vector \((1, 1)\) in \(W^+\) goes from column \(p - 1\) to column 1, and the toroidal vector \((2, 1)\) goes either from column \(p - 2\) to column 1 or from column \(p - 1\) to column 2. In either case, the permutation property of \(W\) forces the toroidal vectors \((1, 1)\) and \((2, 1)\) to share a ‘1’ entry in \(W^+\). Since both vectors have height 1, their two unshared endpoints then lie in the same row of \(W^+\) (in adjacent columns), contradicting the permutation property of \(W\).

To show that \(W\) contains at least one of the vectors \((1, -1)\) and \((2, -1)\), note that the corresponding toroidal vectors in \(W^+\) are \((1, p - 1)\) and \((2, p - 1)\). The result follows from [9, Theorem 4.2] and symmetry. □

**Theorem 4.16.** For \(p \geq 5\), every \(W_1(p, \phi, c)\) Welch Costas array contains a \((1, 1)\) mirror pair or a \((1, 2)\) mirror pair or a \((2, 1)\) mirror pair.

**Proof.** Let \(W\) be a \(W_1(p, \phi, c)\) Welch Costas array with \(p \geq 5\), and suppose that \(W\) contains neither a \((1, 1)\) mirror pair nor a \((2, 1)\) mirror pair. We shall show that \(W\) contains a \((1, 2)\) mirror pair. By Lemma 4.15 and symmetry, we may assume that \(W\) contains the vectors \((1, 1)\) and \((2, -1)\) but not \((1, -1)\) and not \((2, 1)\). Since the array \(W\) is G-symmetric, the vectors \((1, 1)\) and \((2, -1)\) must both cross its vertical bisector. Then by the permutation property, these vectors share an endpoint; their two unshared endpoints lie both in the left half or both in the right half of the array, separated by the vector \((1, -2)\). By G-symmetry, \(W\) therefore contains a \((1, 2)\) mirror pair. □

### 4.3. Mirror pairs in Golomb Costas arrays

In this section, we show the existence of mirror pairs with constrained width and height in a Golomb Costas array \(G\) by using results from [9] on the toroidal vectors of the augmented array \(G^*_1\). In particular, we show that every Golomb Costas array of order at least 7 contains a \((1, h)\)-mirror pair with \(h \in \{1, 2, 3\}\). So every Golomb Costas array of order at least 7 contains a mirror pair of width 1, and therefore a mirror pair of height 1 by Remark 4.5 (since the set of
Golomb Costas arrays is closed under the transpose operation. This shows that the answer to Question 4.4 for Golomb Costas arrays is yes.

If a Costas array does not contain a (1, h)-mirror pair then it does not contain the vector (1, h) or it does not contain the vector (1, −h). For each \( h \in \{1, 2, 3\} \), Lemma 4.17 gives conditions on \( \phi \) and \( \rho \) when a \( G_2(q, \phi, \rho) \) Costas array of sufficient size does not contain the vector (1, h). The result is an excerpt from [4, Theorem 3] and follows the same proof, except that it replaces the argument of [4, Lemma 1] by that of [9, Proposition 3.2] (see also [13, Lemma 82]).

**Lemma 4.17.** [4] \( q \) be a prime power, let \( \phi \) and \( \rho \) be primitive in \( \mathbb{F}_q \) and let \( G \) be the \( G_2(q, \phi, \rho) \) Costas array.

(i) For \( q \geq 4 \), if \( G \) does not contain the vector (1, 1) then \( \rho = \phi \).

(ii) For \( q \geq 5 \), if \( G \) does not contain the vector (1, 2) then \( \rho = \phi^2 \) or \( \phi + \rho = 0 \).

(iii) For \( q \geq 7 \), if \( G \) does not contain the vector (1, 3) then \( \rho = \phi^3 \) or \( \phi^2 + \phi + \rho = 0 \).

**Proof.** Suppose that \( G \) does not contain the vector (1, h) for some \( h \in \{1, 2, 3\} \). Since \( G \) has order \( q - 2 \), we may assume that \( h \leq (q - 2) - 1 \). In the case that the toroidal vector (1, h) is not contained in \( G^*_2 \), we find by [9, Proposition 3.2] that \( \rho = \phi^h \); this is listed as a possible conclusion in each of the cases (i), (ii) and (iii).

Otherwise, the toroidal vector (1, h) is contained in \( G^*_2 \). Let the starting position of this toroidal vector in \( G^*_2 \) be \( (i, j) \). Then by [9, Proposition 3.2],

\[
(1 - \phi^i)(\phi^h - \rho) = \phi^h - 1,
\]

or equivalently, multiplying by \( \phi^{-i} \),

\[
\phi^h - \phi^{-i} + \phi^{-i}\rho - \rho = 0.
\]

(4.2)

Since \( G \) does not contain the vector (1, h) and the last column of \( G^*_2 \) does not contain any 1s, the width 1 toroidal vector (1, h) must cross the horizontal boundary of \( G^*_2 \). Therefore the solution \( i \) to (4.2) satisfies

\[
i < q - 1 < i + h.
\]

(4.3)

For (i), take \( h = 1 \) in (4.3) to show there are no further possible conclusions.

For (ii), take \( h = 2 \) in (4.3) to give \( i = q - 2 \). Then (4.2) gives

\[
(\phi - 1)(\phi + \rho) = 0,
\]

so \( \phi + \rho = 0 \) since \( \phi \) is primitive.

For (iii), take \( h = 3 \) in (4.3) to give \( i = q - 2 \) or \( i = q - 3 \). Either \( i = q - 2 \), and then (4.2) gives

\[
(\phi - 1)(\phi^2 + \phi + \rho) = 0
\]

so that \( \phi^2 + \phi + \rho = 0 \), or \( i = q - 3 \), and then (4.2) gives

\[
(\phi - 1)(\phi^2 + \phi \rho + \rho) = 0
\]

so that \( \phi^2 + \phi \rho + \rho = 0 \). \( \square \)
We now combine conditions from Lemma 4.17 in Theorem 4.18, in order to classify the Golomb Costas arrays that do not contain a \((1,h)\)-mirror pair for all \(h \in \{1,2,3\}\). (We restrict to \(q \geq 7 \) because Golomb Costas arrays of order \(q - 2 \leq 3\) trivially do not contain a mirror pair.) The four exceptional arrays in the statement of Theorem 4.18 form an entire equivalence class under the action of \(D_4\) because the Costas array corresponding to the permutation \([5,3,2,6,1,4]\) is symmetric.

**Theorem 4.18.** The only \(G_2(q, \phi, \rho)\) Golomb Costas arrays of order \(q - 2 \geq 5\) that do not contain a \((1,h)\)-mirror pair for all \(h \in \{1,2,3\}\) are that corresponding to the permutation \([5,3,2,6,1,4]\) and its image under horizontal reflection, vertical reflection and \(180^\circ\) rotation.

**Proof.** Let \(q \geq 7\) be a prime power, let \(\phi\) and \(\rho\) be primitive in \(F_q\), and suppose that the \(G_2(q, \phi, \rho)\) Golomb Costas array \(G\) does not contain a \((1,h)\)-mirror pair for all \(h \in \{1,2,3\}\). Then, for all \(h \in \{1,2,3\}\), \(G\) does not contain the vector \((1,h)\) or does not contain the vector \((1,-h)\).

Lemma 4.17 gives conditions on \(\rho\) and \(\phi\) when \(G\) does not contain the vector \((1,h)\). We can deduce conditions when \(G\) does not contain the vector \((1,-h)\) by replacing \(\rho\) with \(\rho^{-1}\) in Lemma 4.17, since the array \(G_2(q, \phi, \rho^{-1})\) is the reflection in a vertical axis of \(G\). Therefore from Lemma 4.17 we have

\[
\rho = \phi \quad \text{or} \quad \rho = \phi^{-1}
\]

by part (i), and

\[
\rho = \phi^2 \quad \text{or} \quad \phi + \rho = 0 \\
or \quad \rho = \phi^{-2} \quad \text{or} \quad \phi + \rho^{-1} = 0
\]

by part (ii), and

\[
\rho = \phi^3 \quad \text{or} \quad \phi^2 + \phi + \rho = 0 \quad \text{or} \quad \phi^2 + \phi \rho + \rho = 0 \\
or \quad \rho = \phi^{-3} \quad \text{or} \quad \phi^3 + \phi + \rho^{-1} = 0 \quad \text{or} \quad \phi^3 + \phi \rho^{-1} + \rho^{-1} = 0
\]

by part (iii). We will find all arrays corresponding to solutions to (4.5) and (4.6) that satisfy the condition \(\rho = \phi\) of (4.4); arrays corresponding to solutions that satisfy the other condition \(\rho = \phi^{-1}\) of (4.4) then occur as reflections of these arrays in a vertical axis.

We therefore set \(G = G_2(q, \phi, \phi)\), and consider the four cases specified by (4.5) when \(\rho = \phi\).

(A) \(\phi = \phi^2\). This gives \(\phi = 1\) and therefore \(q = 2\) (since \(\phi\) is primitive), contrary to assumption.

(B) \(\phi + \phi = 0\). This forces \(q\) to be a power of 2.

(C) \(\phi = \phi^{-2}\). This gives \(\phi^3 = 1\) and therefore \(q = 2\) or 4, contrary to assumption.

(D) \(\phi + \phi^{-1} = 0\). This gives \(\phi^2 = -1\) and therefore \(q = 5\), contrary to assumption.

We therefore take \(q\) to be a power of 2, as in case (B), and consider the six cases specified by (4.6) when \(\rho = \phi\).
(a) \( \phi = \phi^3 \). This gives \( \phi^2 = 1 \) and therefore \( q = 3 \), contrary to assumption.
(b) \( \phi^2 + \phi + \phi = 0 \). This gives \( \phi^2 = 0 \), a contradiction.
(c) \( \phi^2 + \phi^2 + \phi = 0 \). This gives \( \phi = 0 \), a contradiction.
(d) \( \phi = \phi^3 \). This gives \( \phi^2 = 1 \) and therefore \( q = 2 \), contrary to assumption.
(e) \( \phi^2 + \phi + \phi^{-1} = 0 \). This gives \( \phi^3 + \phi^2 + 1 = 0 \). Since the polynomial \( x^3 + x^2 + 1 \) is irreducible over \( \mathbb{F}_2 \), it is the minimal polynomial of \( \phi \) and so \( q = 8 \).
Then \( G \) is the \( G_2(8, \phi, \phi) \) Costas array where \( \phi^3 + \phi^2 + 1 = 0 \), which corresponds to the permutation \([5, 3, 2, 6, 1, 4]\).
(f) \( \phi^2 + 1 + \phi^{-1} = 0 \). This occurs by replacing \( \phi \) in case (e) with \( \phi^{-1} \). Since the array \( G_2(q, \phi^{-1}, \phi^{-1}) \) is the \( 180^\circ \) rotation of \( G_2(q, \phi, \phi) \), the array arising from this case is the \( 180^\circ \) rotation of that arising from case (e).

The arrays corresponding to the solutions of (4.4), (4.5) and (4.6) are therefore \( G_2(8, \phi, \phi) \), corresponding to the permutation \([5, 3, 2, 6, 1, 4]\), together with its image under horizontal reflection, vertical reflection and \( 180^\circ \) rotation. It is easily verified that the array \( G_2(8, \phi, \phi) \) does not contain a \((1, h)\) mirror pair for all \( h \in \{1, 2, 3\} \).

\[\square\]

References

[8] A. Freedman and N. Levanon, Any two \( N \times N \) Costas arrays must have at least one common ambiguity sidelobe if \( N > 3 \) — a proof, Proceedings of the IEEE 73 (1985), 1530–1531.