A general bound on the weight choosability number of a graph

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Motivation

One of the first “facts” of graph theory: every graph on at least 2 vertices has two vertices with the same degree.

This is not true for multigraphs. Multigraphs where \( d(u) \neq d(v) \) for every distinct pair of vertices \( u, v \) are called \textbf{irregular}. 
Motivation

Question

Can we always transform a simple graph $G$ into an irregular multigraph $M$ by replicating edges?

Answer

Yes, as long as no connected component is isomorphic to $K_2$ (a.k.a. $G$ is nice).

Let $E(G) = \{e_1, e_2, \ldots, e_m\}$. Replace $e_i$ with $2i$ parallel edges.
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Instead, we could weight the edges of \( G \) from \( \{1, 2, \ldots, k\} \) and ask that all vertices have distinct sums of incident edge weights. Such an edge \( k \)-weighting is called irregular.

Question (rephrased)

Given a graph \( G \), what is the minimum value of \( k \) such that there exists an irregular \( k \)-weighting of \( E(G) \)?
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![Graph](image)

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One natural variant of the problem: weight the edges and ask only for adjacent vertices to receive distinct sums of incident edge weights.
Vertex-colouring edge-weightings

An edge $k$-weighting $w$ of a graph $G$ properly colours $V(G)$ by sums if, for each $uv \in E(G)$,

$$\sum_{e \ni u} w(e) \neq \sum_{e \ni v} w(e).$$
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Let $\chi^e(G)$ denote the smallest $k$ for which $G$ has an edge $k$-weighting that properly colours $V(G)$ by sums. Call $\chi^e(G)$ the edge-weight chromatic number of $G$. 
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$$G : \quad \begin{array}{c}
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
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G: 

```
1   1   1   1   2
```

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\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}$

```
1   1   2
```

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

```
1   1   1
```

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
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1
```

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\[ G : \begin{array}{c}
1 & 1 & 3 & 1 & 1 & 2 & 1 & 4 & 2 & 1 & 3
\end{array} \] 

$\Rightarrow \chi^e(G) = 2$
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1 & 1 & 1 & 1 & 1 \\
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$\implies \chi^e(G) = 2$
1-2-3 Conjecture (Karoński, Łuczak, Thomason; ’04)

If $G$ is a nice graph, then $\chi^e(G) \leq 3$. 
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Theorem (Kalkowski, Karoński, Pfender; ’10)
If \( G \) is a nice graph, then \( \chi^e(G) \leq 5 \).

▶ Improves on previous best upper bounds:
   ▶ 30 (Addario-Berry, Dalal, McDiarmid, Reed, Thomason ’07)
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Theorem (Addario-Berry, Dalal, Reed; ’08)

If $G$ is a random graph from $G_{n,p}$ with constant $p \in (0, 1)$, then a.a.s. $\chi^e(G) \leq 2$. 
List 1-2-3 Conjecture

Each edge is given a list of \( k \) arbitrary reals, and we choose our edge weights from those lists. Let \( \text{ch}(G) \) denote the smallest \( k \) for which \( G \) has an edge weighting which properly colours \( V(G) \) by sums for any assignment of lists of size \( k \) to the edges. Call \( \text{ch}(G) \) the weight choosability number of \( G \).

List 1-2-3 Conjecture (Bartnicki, Grytczuk, Niwcyk; '09) If \( G \) is a nice graph, then \( \text{ch}(G) \leq 3 \).

▶ Little known, aside from some special classes

Theorem (S. '12+) If \( G \) is a nice \( d \)-degenerate graph, then \( \text{ch}(G) \leq \Delta(G) + d + 1 \).
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A colouring polynomial...

Let $G$ be a nice graph. For each $e \in E(G)$, assign $e$ a variable $x_e$.

Let $D$ be an arbitrary orientation of $G$, and let $P_D$ be the following polynomial:

$$P_D := \prod_{(u, v) \in A(D)} (\sum_{e \ni v} x_e - \sum_{e \ni u} x_e).$$

An example:

$$e_2 e_3 e_4 \quad e_1 \quad P_D = (x_2 + x_4)(x_3 - x_1 - x_4)(x_4 - x_2)(x_1 + x_2 - x_3).$$

Key fact: Any values for the $x_e$'s which make $P_D \neq 0$, also give an edge weighting which colours $V(G)$ by sums.
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![Graph diagram]

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**Key fact:** Any values for the $x_e$’s which make $P_D \neq 0$, also give an edge weighting which colours $V(G)$ by sums.
Let $f(x_1,\ldots,x_n)$ be a polynomial over a field $F$. Let $\prod_{i=1}^n x_{t_i}$ be a term of $f$ with maximum total degree and nonzero coefficient. If $S_1,\ldots,S_n$ are subsets of $F$ with $|S_i| > t_i$, then there are values $s_1 \in S_1, s_2 \in S_2,\ldots, s_n \in S_n$ so that $f(s_1,\ldots,s_n) \neq 0$.

For example, consider the polynomial from the previous slide:

\[
P(x_1) = (x_2 + x_4)(x_3 - x_1 - x_4)(x_4 - x_2)(x_1 + x_2 - x_3)
\]

If you have 3 choices for $x_1$, 3 choices for $x_2$, 1 choice each for $x_3$ and $x_4$, then some set of choices makes $P \neq 0$.
A Combinatorial Nullstellensatz approach

Combinatorial Nullstellensatz (Alon ’99)

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- \( \text{ch}^e(G) \leq 3 \) for the corresponding graph.
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How to find a term with nonzero coefficient?
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\[
M_D = \begin{pmatrix}
x_1 & x_2 & x_3 & x_4 \\
e_1 & 0 & 1 & 0 & 1 \\
e_2 & -1 & 0 & 1 & -1 \\
e_3 & 0 & -1 & 0 & 1 \\
e_4 & 1 & 1 & -1 & 0
\end{pmatrix}
\]

\[
\Rightarrow \text{per}(M_D) \neq 0
\]

\[
\Rightarrow (x_2 - x_4)(x_3 - x_1 - x_4)(x_4 - x_2)\neq 0
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- matrix permanents (determinants without the ±1 factors)
- approach was given by Bartnicki, Grytczuk, and Niwcyk ('09)

\[ P_D = (x_2 + x_4)(x_3 - x_1 - x_4)(x_4 - x_2)(x_1 + x_2 - x_3) \]

\[ M_D = \begin{pmatrix} x_1 & x_2 \\ e_1 & 0 & 1 \\ e_2 & -1 & 0 \\ e_3 & 0 & -1 \\ e_4 & 1 & 1 \end{pmatrix} \]

⇒ \[ \text{per}(M_D) \neq 0 \]

⇒ \[ \text{ch}(G) \leq 3 \]
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\[ \Rightarrow \text{per}(M_D) \neq 0 \Rightarrow [x_2 1 x_2] P_D \neq 0 \Rightarrow \text{ch}(G) \leq 3 \]
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\[ L = \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & 0 & -1 & 0 \\ x_2 & 1 & 0 & -1 \\ x_1 & 1 & 1 & 1 \end{pmatrix} \]
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$$PD = (x_2 + x_4)(x_3 - x_1 - x_4)(x_4 - x_2)(x_1 + x_2 - x_3)$$

$$L = \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_1 & x_2 & x_2 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow \text{per}(L) \neq 0 \Rightarrow \text{ch}(G) \leq 3$$
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Through some technical lemmas, we show the following:
New results

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Theorem (S. ’12+)

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*Let $G$ be a nice $d$-degenerate graph with $m$ edges, $D$ an orientation of $G$,*
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Theorem (S. ’12+)

Let $G$ be a nice $d$-degenerate graph with $m$ edges, $D$ an orientation of $G$, and $M_D$ the coefficient matrix arising from the polynomial $P_D$. 
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**Theorem (S. ’12+)**

Let $G$ be a nice $d$-degenerate graph with $m$ edges, $D$ an orientation of $G$, and $M_D$ the coefficient matrix arising from the polynomial $P_D$. The matrix consisting of $\Delta(G) + d$ concatenated copies of $M_D$ contains an $m \times m$ submatrix with nonzero permanent.
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The proof exploits a very simple structure – an induced $P_3$ – to obtain an inductive argument.
New results

$G$ a nice $d$-degenerate graph with $m$ edges:

$\implies$ there exists an $m \times m$ matrix with nonzero permanent consisting only of columns from $M_D$, none repeated more than $\Delta(G) + d$ times.
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$\iff$ there is a term in $P_D$ with nonzero coefficient whose largest exponent is at most $\Delta(G) + d$.

$\iff$ by the Combinatorial Nullstellensatz...

Theorem (S. ’12+)

If $G$ is a nice $d$-degenerate graph, then $\text{ch}^e(G) \leq \Delta(G) + d + 1$. 
Current best result on \((1, l)\)-weight choosability:

**Theorem (Pan, Yang ’12)**

*Every nice \(d\)-degenerate graph is \((1, 2d)\)-weight choosable.*
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**Problems**

1. *Is \(\text{ch}^e(G)\) bounded by a constant?*
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For more information:
B. Seamone, “The 1-2-3 Conjecture and related problems: a survey” (arXiv.org)
Thank you