On the directed Oberwolfach Problem with equal cycle lengths

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Outline

- Introduction: The directed Oberwolfach Problem.
- Main results.
- Terminology.
- Tools.
- Ideas from proofs.
Resolvable directed cycle systems
— the directed Oberwolfach Problem with equal cycle lengths

- Directed Oberwolfach Problem with equal cycle lengths:
  Determine the necessary and sufficient conditions on \( n \) and \( m \) for there to exist a decomposition of \( K_n^* \) into spanning subdigraphs, each a disjoint union of directed \( m \)-cycles (that is, a \( RCS^*(m, n) \)).

- Obvious necessary condition: \( m | n \).

- Previous results:

  **Theorem (Bermond, Germa, Sotteau, 1979)**
  
  There exists a \( RCS^*(3, n) \) if and only if \( 3 | n \) and \( n \neq 6 \).

  **Theorem (Bennett, Zhang, 1990)**
  
  There exists a \( RCS^*(4, n) \) if and only if \( 4 | n \) and \( n \neq 4 \).
New results

Theorem (Burgess, Francetić, Niesink, Šajna)

There exist the following:

- a $RCS^*(m, \alpha m)$ for odd $m \geq 5$ and odd $\alpha$;
- a $RCS^*(m, \alpha m)$ for even $m \geq 6$ and all $\alpha \geq 2$;
- a $RCS^*(m, \alpha m)$ for odd $m$, $7 \leq m \leq 49$, $3 \nmid m$, and all even $\alpha$;
- a $RCS^*(m, \alpha m)$ for $m = 9, 15, 21$, and all even $\alpha$.

Lemma

If there exists a $RCS^*(m, 2m)$, then there exists a $RCS^*(m, \alpha m)$ for all even $\alpha$. 
Basic terminology

- \( G^* \): the (symmetric) digraph obtained from a graph \( G \) by replacing each edge \( uv \) with the two arcs \((u, v)\) and \((v, u)\)
- \( K^*_n, K^*_m \)
- \( C_m \) and \( \vec{C}_m \): cycle and directed cycle of length \( m \)
- **Wreath product** \( G \star H \) of (di)graphs \( G \) and \( H \): obtained from \( G \) by replacing every vertex \( u \) of \( G \) with a copy \( H_u \) of \( H \), and for each edge \( uv \) (arc \((u, v)\)) of \( G \), inserting an edge (arc) from every vertex of \( H_u \) to every vertex of \( H_v \)
Terminology

- **Decomposition** $G = H_1 \oplus H_2 \oplus \ldots \oplus H_k$: partition of $E(G)$ into $E(H_1), E(H_2), \ldots, E(H_k)$, for subgraphs $H_1, H_2, \ldots, H_k$ of $G$

- **$H$-decomposition** of a graph $G$: decomposition of $G$ into copies of a subgraph $H$

- **Resolution class:**
  a subset $\{H_{i_1}, H_{i_2}, \ldots, H_{i_t}\}$ of a decomposition $\mathcal{D} = \{H_1, H_2, \ldots, H_k\}$ of $G$ such that $\{V(H_{i_1}), V(H_{i_2}), \ldots, V(H_{i_t})\}$ is a partition of $V(G)$

- **Resolvable decomposition:**
  a decomposition that can be partitioned into resolution classes

- **$RCS(m, G)$**: resolvable $m$-cycle decomposition of a graph $G$

- **$CS^*(m, D)$**: directed $m$-cycle decomposition of a digraph $D$

- **$RCS^*(m, D)$**: resolvable dir. $m$-cycle decomposition of a digraph $D$
Tools: previous results

Theorem (Alspach, Schellenberg, Stinson, Wagner)

There exists a $RCS(m, K_n)$ if and only if $n$ is odd and $m|n$.

Theorem (Alspach, Jordon, Šajna, Verrall)

There exists a $CS^*(m, K_n^*)$ if and only if $m|n(n - 1)$ and $(m, n) \notin \{(4, 4), (3, 6), (6, 6)\}$.

Theorem (Liu)

There exists a $RCS(m, K_n \star \bar{K}_t)$ if and only if $m|nt$, $t(n - 1)$ is even, $m$ is even if $n = 2$, and $(m, n, t) \notin \{(3, 3, 2), (3, 3, 6), (3, 6, 2), (6, 2, 6)\}$.
For even $m \geq 4$, there exist a $CS^*(m, K_{m^2/2}, m)$ and $RCS^*(m, K_{m,m})$. 
Let $m \geq 4$ be even and $\alpha \geq 3$.
There exists a $RCS^*(m, C^*_\alpha \star \overline{K}_m)$.
Case $\alpha$ odd:
Let $m \geq 4$ be even and $\alpha \geq 2$.

Case $\alpha$ even: there exists a 1-factorization of $K_{\alpha}$.

Case $\alpha$ odd: there exists a $RCS(\alpha, K_{\alpha})$.

Hence there exists a $RCS^*(m, K_{\alpha}^* \star \bar{K}_m)$. 

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On the directed Oberwolfach Problem

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Proof ideas: Case \( \alpha \) and \( m \) both odd, or \( m \geq 8 \) even.

Case \( \alpha \) and \( m \) both odd.
- By [ASSW], there exists a \( RCS(m, K_{\alpha m}) \).
- Direct each cycle in this decomposition once in each possible direction to obtain a \( RCS^*(m, K_{\alpha m}^*) \).

Case \( m \geq 8 \) even.
- Decompose \( K_{\alpha m}^* = K_\alpha^* \star K_m^* \oplus K_{\alpha}^* \star \bar{K}_m \).
- There exists a \( CS^*(m, K_m^*) \) by [AJŠV].
- Hence there exists a \( RCS^*(m, K_\alpha^* \star K_m^*) \).
- There exists a \( RCS^*(m, K_\alpha^* \star \bar{K}_m) \) as seen before (also by [Liu]).
Proof ideas: Case $m = 6$.

Challenge: a $CS^*(6, K_6^*)$ does not exist.

**Case $\alpha \geq 2$ even.**

- Decompose $K_{6,\alpha} = \bar{K}_{\frac{\alpha}{2}} \ast K_{12}^* \oplus K_{\frac{\alpha}{2}}^* \ast \bar{K}_{12}$.
- There exists a $RCS^*(6, K_{\frac{\alpha}{2}}^* \ast \bar{K}_{12})$ by [Liu].
- There exists a $RCS^*(6, K_{12}^*)$ (shown on the next page).
Proof ideas: Case $m = 6$ — continued.

A $\text{RCS}^{*}(6, K_{12}^*)$:

+ mirror image

$+ 2$ more resolution classes
Proof ideas: Case $m = 6$ — continued.

Case $\alpha \geq 3$ odd.

- There exists a $CS(\alpha, K_{\alpha})$.
- Decompose $K_{6\alpha}$ into $C^{*}_{\alpha} \star K_{6}^{*}$ and $\frac{\alpha-1}{2}$ copies of $C^{*}_{\alpha} \star \bar{K}_{6}$.
- There exists a $RCS^{*}(6, C^{*}_{\alpha} \star K_{6}^{*})$ (special construction).
- There exists a $RCS^{*}(6, C^{*}_{\alpha} \star \bar{K}_{6})$ as seen.
Proof ideas: Case $\alpha$ even, $m$ odd.

- Decompose $K_{\alpha m}^* = \bar{K}_{\alpha/2}^* \ast K_{2m}^* \oplus K_{\alpha/2}^* \ast \bar{K}_{2m}$.
- By [Liu], there exists a $RCS(m, K_{\alpha/2}^* \ast \bar{K}_{2m})$.
- Hence, there exists a $RCS^*(m, K_{\alpha/2}^* \ast \bar{K}_{2m})$.
- Then the existence of a $RCS^*(m, K_{2m}^*)$ implies the existence of a $RCS^*(m, K_{\alpha m}^*)$. 

![Diagram of proof ideas](image-url)
What we know about $RCS^*(m, 2m)$

**Proposition**

$RCS^*(m, 2m)$ exists for

- odd $m$, $7 \leq m \leq 49$, $3 \nmid m$;
- $m = 9, 15, 21$. 

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Proof ideas: $m = 5$

There exist a $RCS^*(5, K_{10}^*)$: 

![Diagram of a directed graph with 10 vertices and directed edges connecting each vertex to every other vertex. The graph is symmetric and complete.]
Proof ideas: A $\text{RCS}^*(m, 2m)$ for $m \equiv 1$ or $5 \pmod{6}$, $7 \leq m \leq 49$. 
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![Diagram showing directed paths](Image)
Proof ideas: A $RCS^*(m, 2m)$ for $m \equiv 1$ or $5 \pmod{6}$, $7 \leq m \leq 49$. 

Left:  

Right:
Proof ideas: A $RCS^*(m, 2m)$ for $m \equiv 1$ or 5 (mod 6), $7 \leq m \leq 49$. 

Left

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It remains to decompose the digraph on the right into:
- vertex-disjoint dir. \((5, 2)\)-path of length 2 and \((3, 4)\)-path of length 3, and
- directed Hamilton cycles.
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- vertex-disjoint dir. $(5, 2)$-path of length 2 and $(3, 4)$-path of length 3, and
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Thank you!