Weak Near-Unanimity Functions and NP-Completeness Proofs

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Outline

- The Homomorphism Problem.
  - Complexity in the undirected case: Hell and Nešetřil (1990).
  - Complexity in the directed case: Unknown, only special cases e.g. semi-complete digraphs (Bang-Jensen, Hell and MacGillivray, 1988), smooth digraphs (Barto, Kozik and Niven 2008).
- Weak Near-Unanimity Functions and NP-completeness.
  - Conjecture of Bulatov, Jeavons and Krokhin.
  - Gather evidence for this conjecture:
    - Some polynomial cases: $\underline{X}$, C_k-extended $\underline{X}$, graft extension.
    - NP-completeness: Indicator construction, vertex and arc sub-indicators.
- WNUFs/no WNUFs for undirected graphs, semi-complete digraphs, and vertex transitive digraphs.

The Homomorphism Problem

- Let G and H be digraphs. A homomorphism from G to H is a mapping $f : V(G) \longrightarrow V(H)$ such that $xy \in A(G)$ implies that $f(x)f(y) \in A(H)$. The existence of such a homomorphism is denoted by $G \longrightarrow H$.
- The homomorphism problem for the digraph H is the problem of deciding for a given input G whether $G \longrightarrow H$. This is also known as the H-colouring problem, and is denoted by HOM_H.

Complexity in the Undirected Case

In the undirected case there is a dichotomy:

Theorem (Hell and Nešetřil 1990)

Let H be a graph with loops allowed.

- *If H is bipartite or contains a loop, then the H-colouring problem has a polynomial time algorithm.*
- *Otherwise the H-colouring problem is NP-complete.*

Complexity in the Directed Case

Only known in special cases:

Theorem (Bang-Jensen, Hell and MacGillivray 1988)

Let H be a semi-complete digraph.

- *If H contains at most one directed cycle, then H-colouring is polynomial time solvable.*
- *Otherwise H-colouring is NP-complete.*

Theorem (Barto, Kozik and Niven 2008)

Let H be a digraph with no sources and no sinks.

- *If the core of H is a directed cycle, then HOM_H is polynomial.*
- *If the core of H is not a directed cycle, then HOM_H is NP-complete.*

Weak Near-Unanimity Functions

The result by Barto, Kozik and Niven uses tools from universal algebra:

Theorem (Bulatov, Jeavons and Krokhin; Larose and Zádori; Maróti and McKenzie)

If the digraph H does not admit a weak near unanimity function, then HOM_H is NP-complete.

The existence of such a function is conjectured by Bulatov, Jeavons and Krokhin to determine the complexity of HOM_H exactly:

Conjecture

If the digraph H admits a WNUF, then HOM_H is polynomial, otherwise (if H does not admit a WNUF) HOM_H is NP-complete.

Weak Near-Unanimity Functions

A weak near-unanimity function f of arity k (WNUF_k) on the digraph H is

- a polymorphism: $f : H^k \to H$,

  $$V(H^k) = \overbrace{V(H) \times V(H) \times \cdots \times V(H)}^{k}$$
  $$(x_1, x_2, \ldots, x_k) \to (y_1, y_2, \ldots, y_k) \text{ iff } x_i \to y_i,\ 1 \le i \le k,$$

- idempotent: $f(x, x, \ldots, x) = x$,

- weakly nearly-unanimous:

  $$f(y, x, x, \ldots, x, x) = f(x, y, x, \ldots, x, x) = f(x, x, y, \ldots, x, x) = \cdots = f(x, x, x, \ldots, x, y).$$

Conjecture of Bulatov, Jeavons and Krokhin

Conjecture (Bulatov, Jeavons and Krokhin)

If H admits a WNUF, *then* HOM_H *is polynomial time solvable. Otherwise* HOM_H *is NP-complete (this part is known).*

To gather evidence for this conjecture we could:

- Show that known algorithmic methods $\Rightarrow$ WNUFs.
- Try to show that digraphs H for which HOM_H is NP-complete, have no WNUF. Try to prove "no-WNUF" theorems.
- Prove it for various graph families.
  - Good candidates are ones where a dichotomy is known.

Theorem (Gutjahr, Woeginger and Welzl 1992)

Let H be a digraph such that

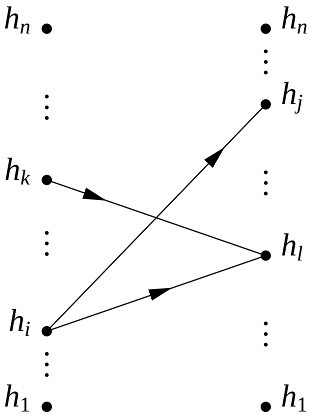
- *H has an $\underline{X}$-enumeration, or*
- *H has the C_k-extended $\underline{X}$ property, or*
- *H is an instance of the graft extension.*

Then HOM_H is polynomial time solvable.

The goal here is to show that if a digraph H has any of these properties, then it also has a WNUF.

The $\underline{X}$-enumeration

Let $\{h_1, h_2, \ldots, h_n\}$ be an enumeration of the vertices of a digraph H. This enumeration is said to satisfy the $\underline{X}$ property if $h_i h_j$, $h_k h_l \in A(H)$, implies that $\min(h_i, h_k)\min(h_j, h_l) \in A(H)$, where the minimum is taken with respect to the enumeration of $V(H)$.



Theorem

If H has an $\underline{X}$-enumeration, then H has a WNUF$_2$.

Proof: Define $f : H^2 \to H$ by $f(x_1, x_2) = \min\{x_1, x_2\}$. ∎

A converse is also true.

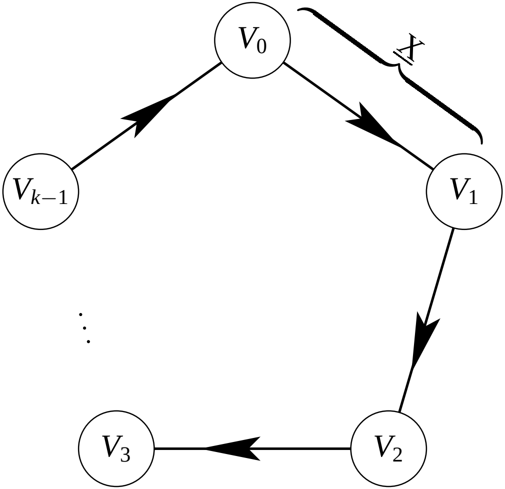
Theorem

If there is an enumeration of $V(H)$ and a WNUF$_2$, $f : H^2 \to H$, such that for all $x_1, x_2 \in V(H)$, $f(x_1, x_2) = \min\{x_1, x_2\}$, then this enumeration is an $\underline{X}$-enumeration.

Proof: Suppose u_1u_2 and $v_1v_2 \in E(H)$. Then $(u_1, v_1)(u_2, v_2) \in E(H \times H)$, so $f(u_1, v_1) = \min\{u_1, v_1\}$ is adjacent to $f(u_2, v_2) = \min\{u_2, v_2\}$ in H. ∎

Arity 2 can be replaced by arity k.

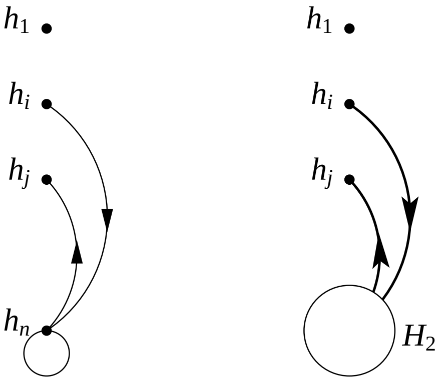
The C_k-extended $\underline{X}$ Property



The $\underline{X}$-graft Extension

- $h_1, h_2, \ldots, h_n$ is an $\underline{\text{X}}$-enumeration H_1.
- H_2 is another digraph.

Form H by replacing h_n by H_2 as in the wreath product. The digraph H is called graft(H_1, H_2).



Theorem

Let H be a digraph such that

- *H has an $\underline{X}$-enumeration, or*
- *H has the C_k-extended $\underline{X}$ property, or*
- $H = graft(H_1, H_2)$, *where H_2 has a $WNUF_k$.*

Then H has a $WNUF_k$.

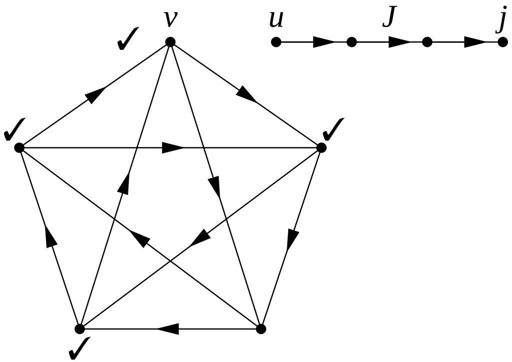
NP-completeness Proofs and WNUFs

Four main tools in proving NP-completeness of HOM_H:

- Direct reduction from SAT or similar.
- The “indicator” construction.
- The “(vertex) sub-indicator” construction.
- The “arc sub-indicator” construction.

For digraphs H in some family, the proofs normally go by induction on $|V(H)|$ (non-existence of minimum counterexample). The direct reductions establish the base cases and the three constructions make the induction work.

Vertex Sub-indicator Example



Identify u and v, retract J back to T_5, and keep track of the images of j.

Vertex Sub-indicator

In general, some designated vertices of the sub-indicator J are identified with some designated vertices of H. Let H^+ be the subgraph of H induced by the images of j under retraction back to H.

Lemma (Hell and Nešetřil 1990)

Let H be a digraph that is a core. If the H^+-colouring problem is NP-complete, then the H-colouring problem is also NP-complete.

Lemma

Let H be a digraph. If H^+ does not have a WNUF_k *for* $k > 1$, *then H does not have a* WNUF *of arity greater than one.*

The proof shows the contrapositive.

There are similar results corresponding to the indicator construction and arc sub-indicator construction.

NP-completeness Proofs and WNUFs

These results give a method for translating certain NP-completeness theorems into no WNUF theorems.

- “NP-complete” $\rightsquigarrow$ “no WNUF.”
- If there is a proof of NP-completeness that depends on base cases $B_1, B_2, \ldots, B_t$ and the application of vertex (arc) sub-indicators and indicators, then there is a proof of “no WNUF” provided one can show that $B_1, B_2, \ldots, B_t$ have no WNUF.

Semi-complete Digraphs and WNUFs

Theorem (Bang-Jensen, Hell and MacGillivray 1988)

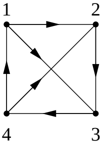
Let H be a semi-complete digraph.

- *If H contains at most one directed cycle, then H-colouring is polynomial time solvable.*
- *Otherwise H-colouring is NP-complete.*

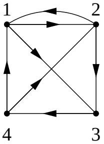
- Acyclic tournaments have an $\underline{X}$-enumeration $\Rightarrow$ have a WNUF.
- Unicyclic tournaments are instances of the graft extension $\Rightarrow$ have a WNUF.

Semi-complete Digraphs With At Least Two Cycles

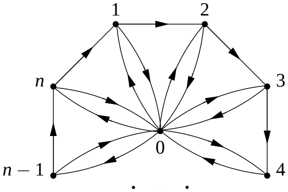
Base Cases:



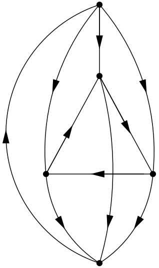
T_4



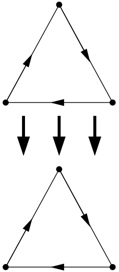
T_4'



W



T_5



T_6

Theorem

Let H be a semi-complete digraph.

- *If H contains at most one directed cycle, then H has a WNUF.*
- *Otherwise H does not admit a WNUF.*

Theorem (Hell and Nešetřil 1990)

Let H be a core.

- *If H is bipartite, then the H-colouring problem has a polynomial time algorithm.*
- *Otherwise the H-colouring problem is NP-complete.*

Theorem

Let H be a core.

- *If $H = K_2$, then H admits a NUF$_3$.*
- *If H is non-bipartite, then H does not admit a WNUF.*

Base case is K_3.

Vertex Transitive Digraphs and WNUFs

Theorem (MacGillivray 1991)

Let H be a vertex transitive digraph that is also a core.

- *If $H = C_k$, HOM_H is polynomial.*
- *Otherwise, HOM_H is NP-complete.*

Theorem

Let H be a vertex transitive digraph that is also a core.

- *If $H = C_k$, then H admits a NUF_3.*
- *Otherwise, H does not admit a WNUF.*

Base Cases: Undirected non-bipartite graphs.

Thank you.
Questions?